

LEFSCHETZ TYPE FORMULAS FOR DG-CATEGORIES

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ABSTRACT. We prove an analog of the holomorphic Lefschetz formula for endofunctors of smooth compact dg-categories. We deduce from it a generalization of the Lefschetz formula of V. Lunts [17] that takes the form of a reciprocity law for a pair of commuting endofunctors. As an application, we prove a version of Lefschetz formula proposed by Frenkel and Ngô in [11]. Also, we compute explicitly the ingredients of the holomorphic Lefschetz formula for the dg-category of matrix factorizations of an isolated singularity.

The derived categories of coherent sheaves and their dg-enhancements have been crucial in some recent developments in algebraic geometry. Among these is the new point of view on the circle of ideas related to Chern characters and Hirzebruch-Riemann-Roch theorem, started in the work of Markarian [18] and continued in [19], [5], [6], [25], [27], [22]. Namely, as shown by Shklyarov in [27], one can formulate and prove a version of the HRR formula for any smooth and proper dg-category \mathcal{C} , where the cohomology of the variety is replaced by the Hochschild homology $HH_*(\mathcal{C})$. The difficulty of applying Shklyarov's formula is usually in the explicit calculation of the Chern character with values in $HH_*(\mathcal{C})$ and of the canonical pairing on $HH_*(\mathcal{C})$. For example, for derived categories of coherent sheaves on smooth projective varieties this was done in [5] and [25]. The case of the category of matrix factorizations associated with an isolated hypersurface singularity was treated in [22]. On the other hand, Lunts proved in [17] a version of the topological Lefschetz trace formula for endofunctors of dg-categories that reduces to the standard one in the case of the derived category of coherent sheaves. In the present paper we will consider a dg-version of the holomorphic Lefschetz formula (in its algebraic version) and will calculate its ingredients explicitly for the category of matrix factorizations of an isolated hypersurface singularity.

Recall that the classical Lefschetz fixed point formula equates the number of fixed points of an endomorphism $f : M \rightarrow M$ of a compact oriented manifold with the supertrace of the action of f on the cohomology of M (assuming the intersection of the graph of f with the diagonal is transversal). In the case when M is a compact complex manifold and f is holomorphic, the holomorphic Lefschetz formula computes the supertrace of the action of f on $H^*(M, V)$, where V is a holomorphic vector bundle equipped with a map $f^*V \rightarrow V$ (see [1, Thm. 2] and [31]). There are also purely algebraic versions of this theorem (see [9], [3], [21], [13, II.6]).

In the abstract dg-context the role of the pair (M, f) is played by a pair (\mathcal{C}, F) , where \mathcal{C} is a dg-category over a field k and F is an endofunctor of $\text{Per}(\mathcal{C})$, the perfect derived category of \mathcal{C} (see [15]). The cohomology of M is replaced in this context by the Hochschild homology of \mathcal{C} . The classical situation is recovered when \mathcal{C} is a dg-version of the bounded derived category $D^b(M)$ of coherent sheaves on a smooth projective variety M over \mathbb{C} , and F is the pull-back functor with respect to an endomorphism of M . Note that in this case, by the Kostant-Hochschild-Rosenberg theorem, the Hochschild homology $HH_*(\mathcal{C})$ is isomorphic to the usual cohomology $H^*(M, \mathbb{C})$. The assumption of M being smooth and compact has a well-known analog for dg-categories. In addition we assume the existence of a generator and consider only endofunctors induced by a kernel in $\text{Per}(\mathcal{C}^{op} \otimes \mathcal{C})$ (i.e., F is a *tensor functor* in the terminology of [15]).

To extend the setup of the holomorphic Lefschetz formula to an arbitrary dg-category we interpret $H^*(M, V)$ as $\text{Ext}^*(\mathcal{O}_M, V)$ and note that (\mathcal{O}_M, V) can be replaced by any pair of finite complexes of

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holomorphic vector bundles (interacting with f appropriately). Thus, our abstract formula requires as an input a pair of objects $A, B \in \text{Per}(\mathcal{C})$ equipped with morphisms $\alpha : A \rightarrow F(A)$ and $\beta : F(B) \rightarrow B$. There is an induced endomorphism

$$(F, \alpha, \beta)_* : \text{Hom}(A, B) \xrightarrow{F} \text{Hom}(F(A), F(B)) \rightarrow \text{Hom}(A, B), \quad (0.1)$$

where the second arrow is given by pre-composing with α and post-composing with β , and our formula reads as

$$\text{str}((F, \alpha, \beta)_*, \text{Hom}(A, B)) = \langle \tau_F^A(\alpha), \tau_G^B(\tilde{\beta}) \rangle_{F, G}, \quad (0.2)$$

where $G : \text{Per}(\mathcal{C}) \rightarrow \text{Per}(\mathcal{C})$ is the right adjoint functor to F , $\tilde{\beta} : B \rightarrow G(B)$ is induced by β ,

$$\tau_F^A : \text{Hom}(A, F(A)) \rightarrow \text{Tr}_{\mathcal{C}}(F)$$

is a certain *generalized boundary bulk map* taking values in a graded vector space $\text{Tr}_{\mathcal{C}}(F)$ associated with F , and $\langle ?, ? \rangle_{F, G}$ is a canonical perfect pairing between $\text{Tr}_{\mathcal{C}}(F)$ and $\text{Tr}_{\mathcal{C}}(G)$ that we will define. More precisely, here $\text{Tr}_{\mathcal{C}}$ is the functor on $\mathcal{C} - \mathcal{C}$ -bimodules given by tensor multiplication with the diagonal bimodule, so $\text{Tr}_{\mathcal{C}}(F)$ is just the Hochschild homology with values in the bimodule corresponding to F . In the case when $F = \text{Id}_{\mathcal{C}}$ the space $\text{Tr}_{\mathcal{C}}(F)$ becomes $HH_*(\mathcal{C})$ and the formula (0.2) is exactly the generalized abstract Hirzebruch-Riemann-Roch Theorem [22, Thm. 3.1] (a similar property is called “Baggy Cardy Condition” in [6]).

The data consisting of the spaces $\text{Tr}_{\mathcal{C}}(F)$ and $\text{Tr}_{\mathcal{C}}(G)$ and the pairing between them should be thought of as an abstract replacement of the derived intersection of the graph of an endomorphism $f : M \rightarrow M$ with the diagonal. In the geometric situation (assuming that the intersection is transversal) the spaces $\text{Tr}_{\mathcal{C}}(F)$ and $\text{Tr}_{\mathcal{C}}(G)$ have bases numbered by the fixed points of f , and the pairing $\langle ?, ? \rangle_{F, G}$ is diagonal with respect to these bases. It is not clear whether one can formulate an analog of transversality in the abstract situation.

The dg-version of the topological Lefschetz formula, recently established by Lunts [17], has as an input a pair (\mathcal{C}, F) as above and states the equality

$$\text{str}(F_*, HH_*(\mathcal{C})) = \text{sdim } \text{Tr}_{\mathcal{C}}(F) \quad (0.3)$$

in the ground field k , where F_* is the map on Hochschild homology induced by F and in the right-hand side we use the superdimension of a graded vector space defined by $\text{sdim}(V) = \dim(V^{\text{ev}}) - \dim(V^{\text{odd}})$.

Using formula (0.2) we establish a certain generalization of (0.3) that we call *Lefschetz reciprocity*. Namely, assume that we are given a pair of tensor endofunctors of $\text{Per}(\mathcal{C})$, F and Ψ , together with a morphism

$$f : F \circ \Psi \rightarrow \Psi \circ F.$$

Let G be the right adjoint to F . Then f induces a morphism $\psi : \Psi \circ G \rightarrow G \circ \Psi$. Also, we have the induced map

$$(F, f)_* : \text{Tr}_{\mathcal{C}}(\Psi) \rightarrow \text{Tr}_{\mathcal{C}}(G \circ F \circ \Psi) \xrightarrow{G \circ f \circ F} \text{Tr}_{\mathcal{C}}(G \circ \Psi \circ F) \xrightarrow{\sim} \text{Tr}_{\mathcal{C}}(F \circ G \circ \Psi) \rightarrow \text{Tr}_{\mathcal{C}}(\Psi),$$

where the third arrow uses the key property that one can switch the order of composition under Tr (see (1.10)). Similarly, we have the map $(\Psi, \psi)_* : \text{Tr}_{\mathcal{C}}(G) \rightarrow \text{Tr}_{\mathcal{C}}(G)$. We prove the following reciprocity relation:

$$\text{str}((F, f)_*, \text{Tr}(\Psi)) = \text{str}((\Psi, \psi)_*, \text{Tr}(G)). \quad (0.4)$$

The idea of the proof is to apply (0.2) to a certain endofunctor of $\text{Per}(\mathcal{C}^{\text{op}} \otimes \mathcal{C})$ induced by F . Note that (0.3) is a particular case of (0.4): one should take Ψ to be the identity functor.

In the case when F is an autoequivalence, we deduce from (0.4) a Lefschetz type formula for the action of an autoequivalence F of $\text{Per}(\mathcal{C})$ on the Hochschild *cohomology* of \mathcal{C} . Namely, we show that its supertrace is equal to the supertrace of the automorphism of the space $\text{Tr}(F)$ induced by the inverse Serre functor (see Section 3.3).

There are two important classes of examples in which the abstract formulas (0.2) and (0.4) can be made explicit: the categories of coherent sheaves on smooth projective varieties and the categories of matrix factorizations of a potential with an isolated singularity. Note that in [17] the formula (0.3) is rewritten in explicit terms for the case of coherent sheaves (see also [7]), while in [22] the case $F = \text{Id}_{\mathcal{C}}$ of the formula (0.2) is made explicit for matrix factorizations. In Section 3.4 we extend this calculation to the case of an autoequivalence F induced by a diagonal linear map preserving the potential. In the simplest case when the origin is an isolated fixed point of this map we get the following result.

Theorem 0.1. *Let $t = (t_1, \dots, t_n) \in (k^*)^n$ be a diagonal symmetry of an isolated singularity $\mathbf{w}(x_1, \dots, x_n) \in k[[x_1, \dots, x_n]]$, such that $t_i \neq 1$ for all i (i.e., the origin is an isolated fixed point of t). Then for any pair of matrix factorizations of \mathbf{w} , \bar{A} and \bar{B} , equipped with closed morphisms $\alpha : \bar{A} \rightarrow t^* \bar{A}$ and $\beta : t^* \bar{B} \rightarrow \bar{B}$, one has*

$$\text{str}((t^*, \alpha, \beta)_*, \text{Hom}(\bar{A}, \bar{B})) = \text{str}(\alpha|_0, \bar{A}|_0) \cdot \text{str}(\beta|_0, \bar{B}|_0) \cdot \prod_{i=1}^n (1 - t_i)^{-1}.$$

Finally, we apply the formula (0.4) in the context of the derived category of coherent sheaves on a smooth projective variety to prove a version of holomorphic Lefschetz formula conjectured by Frenkel and Ngô in [11, Conj. 6.2] (see Theorem 3.6). Presumably a similar proof should work in a more general setting of derived stacks, as envisioned in [11].

Convention. In various maps involving tensor products the Koszul sign rule is tacitly assumed. We work over a fixed ground field k . Whenever we talk about varieties over k we assume that k is algebraically closed. For a morphism of schemes f we denote by f^* and f_* the corresponding derived functors.

1. KERNELS AND TRACES

1.1. Kernels, duals and tensor products. We refer to [14], [15], [29] and [30] for results about dg-modules over dg-categories. For a dg-category \mathcal{C} we denote by $\text{Per}(\mathcal{C})$ the derived category of perfect \mathcal{C}^{op} -modules, and by $\text{Per}_{dg}(\mathcal{C})$ its dg-version, so that $\text{Per}(\mathcal{C})$ is the homotopy category of $\text{Per}_{dg}(\mathcal{C})$. For $M, N \in \text{Per}(\mathcal{C})$ we denote by $\text{Hom}(M, N) = \text{Hom}_{\mathcal{C}^{op}}(M, N)$ the morphism spaces in $\text{Per}(\mathcal{C})$. Note that these are given by the cohomology of the complexes $\text{Hom}_{\mathcal{C}^{op}\text{-mod}}(M, N)$. In this paper we consider only dg-categories \mathcal{C} such that $\text{Per}_{dg}(\mathcal{C})$ is saturated, i.e., smooth, compact and has a generator. This implies that the spaces $\text{Hom}(M, N)$ are finite-dimensional.

Every dg-functor $F : \text{Per}_{dg}(\mathcal{C}) \rightarrow \text{Per}_{dg}(\mathcal{D})$ (up to an appropriate equivalence) can be given by a kernel $K \in \text{Per}(\mathcal{C}^{op} \otimes \mathcal{D})$, so that

$$F = T_K : M \mapsto M \otimes_{\mathcal{C}}^{\mathbb{L}} K.$$

This correspondence can be formulated as a certain quasi-equivalence of dg-categories (see [30, Sec. 2.2], [15, Thm. 4.6]). In this paper, whenever we talk about a functor $F : \text{Per}(\mathcal{C}) \rightarrow \text{Per}(\mathcal{D})$, we implicitly assume that it is given on a dg-level, and hence is induced by a kernel. Sometimes, we denote the corresponding kernel in $\text{Per}(\mathcal{C}^{op} \otimes \mathcal{D})$ also by F . We denote by $\Delta_{\mathcal{C}} \in \text{Per}(\mathcal{C}^{op} \otimes \mathcal{C})$ the diagonal kernel representing the identity functor on $\text{Per}(\mathcal{C})$.

The composition of dg-functors $F_1 \circ F_2$, where F_2 is given by a kernel $K_2 \in \text{Per}(\mathcal{C}^{op} \otimes \mathcal{D})$ and F_1 is given by a kernel $K_1 \in \text{Per}(\mathcal{D}^{op} \otimes \mathcal{E})$, corresponds to the kernel $K_2 \otimes_{\mathcal{D}}^{\mathbb{L}} K_1 \in \text{Per}(\mathcal{C}^{op} \otimes \mathcal{E})$. Sometimes, when K_i are implicit, we denote this kernel simply by $F_1 \circ F_2$.

Recall that we have a natural duality

$$\text{Per}(\mathcal{C})^{op} \xrightarrow{\sim} \text{Per}(\mathcal{C}^{op}) : M \mapsto M^{\vee}, \quad (1.1)$$

where $M^{\vee}(C) = \text{Hom}_{\mathcal{C}^{op}\text{-mod}}(M, h_C)$ (morphisms in the dg-category of \mathcal{C}^{op} -modules). Here h_C is the representable \mathcal{C}^{op} -module associated with $C \in \mathcal{C}$. For $M \in \text{Per}_{dg}(\mathcal{C})$ and $N \in \text{Per}_{dg}(\mathcal{C}^{op})$ we have a natural quasi-isomorphism

$$M \otimes_{\mathcal{C}}^{\mathbb{L}} N^{\vee} \rightarrow \text{Hom}_{\mathcal{C}^{op}\text{-mod}}(N, M) \quad (1.2)$$

(see [14, 6.2]). Also, we have a natural quasi-isomorphism $M \rightarrow (M^\vee)^\vee$.

For kernels $K_1 \in \text{Per}(\mathcal{C}_1^{op} \otimes \mathcal{D}_1)$ and $K_2 \in \text{Per}(\mathcal{C}_2^{op} \otimes \mathcal{D}_2)$ we set

$$K_1 \square K_2 = (K_1 \otimes K_2) \circ \sigma_{23} \in \text{Per}(\mathcal{C}_1^{op} \otimes \mathcal{C}_2^{op} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2),$$

where σ_{23} is the permutation. We denote the corresponding functor by

$$F_1 \square F_2 : \text{Per}(\mathcal{C}_1 \otimes \mathcal{C}_2) \rightarrow \text{Per}(\mathcal{D}_1 \otimes \mathcal{D}_2),$$

where F_1 and F_2 are functors associated with K_1 and K_2 .

1.2. Adjoint kernels. Let

$$F = T_K : \text{Per}(\mathcal{C}) \rightarrow \text{Per}(\mathcal{D}) : M \mapsto M \otimes_{\mathcal{C}} K$$

be the functor associated with a kernel $K \in \text{Per}(\mathcal{C}^{op} \otimes \mathcal{D})$. We associate with K two other kernels. First, we define $K^T \in \text{Per}(\mathcal{D}^{op} \otimes \mathcal{C})$, the *right dual kernel to K* , by

$$K^T(D, C^\vee) = \text{Hom}_{\mathcal{D}^{op}\text{-mod}}(K(C, ?), h_D),$$

where h_D is the representable right \mathcal{D} -module associated with D . The corresponding functor

$$G = T_{K^T} : \text{Per}(\mathcal{D}) \rightarrow \text{Per}(\mathcal{C})$$

is right adjoint to F (see [14, Sec. 6.2] and [22, Sec. 1.2]). Furthermore, the adjunction morphisms are induced by the canonical morphisms of kernels in $\text{Per}(\mathcal{D}^{op} \otimes \mathcal{D})$ and $\text{Per}(\mathcal{C}^{op} \otimes \mathcal{C})$:

$$K^T \otimes_{\mathcal{C}}^{\mathbb{L}} K \rightarrow \Delta_{\mathcal{D}} \quad \text{and} \quad (1.3)$$

$$\Delta_{\mathcal{C}} \rightarrow K \otimes_{\mathcal{D}}^{\mathbb{L}} K^T \quad (1.4)$$

(in fact, the first morphism has a natural realization in $\text{Per}_{dg}(\mathcal{D}^{op} \otimes \mathcal{D})$, see [22, (1.14)]). In terms of the duality (1.1) we can rewrite the definition of K^T as follows:

$$K^T(?, C^\vee) = K(C, ?)^\vee$$

We also set

$$K' := K \circ \sigma \in \text{Per}(\mathcal{D} \otimes \mathcal{C}^{op}),$$

where σ is the permutation of factors, and denote by $F' : \text{Per}(\mathcal{D}^{op}) \rightarrow \text{Per}(\mathcal{C}^{op})$ the corresponding functor.

Lemma 1.1. *The pair (G', F') is also adjoint. More precisely, there is a natural quasi-isomorphism of kernels $K' \rightarrow (K^T)^{tT}$.*

Proof. The functor G' is given by the kernel $(K^T)' = K^T \circ \sigma \in \text{Per}(\mathcal{C} \otimes \mathcal{D}^{op})$ satisfying

$$(K^T)'(C^\vee, ?) = K(C, ?)^\vee.$$

Hence, the right adjoint to G' is given by the kernel $(K^T)^{tT} \in \text{Per}(\mathcal{D} \otimes \mathcal{C}^{op})$. We have

$$(K^T)^{tT}(?, C) = (K^T)'(C^\vee, ?)^\vee = K(C, ?)^{\vee\vee}$$

and there is a natural quasi-isomorphism $K'(\cdot, C) = K(C, ?) \rightarrow K(C, ?)^{\vee\vee}$. \square

Lemma 1.2. (i) *For $M \in \text{Per}(\mathcal{D})$ and $N \in \text{Per}(\mathcal{C})$ one has a natural isomorphism*

$$\text{Hom}_{\mathcal{D}^{op}}(M, F(N)) \simeq \text{Hom}_{\mathcal{C}}(N^\vee, F'(M^\vee)).$$

(ii) *For $N \in \text{Per}(\mathcal{C})$ there is a natural isomorphism*

$$G'(N^\vee) \simeq F(N)^\vee \quad (1.5)$$

in $\text{Per}(\mathcal{D}^{op})$, so that the following diagram of isomorphisms is commutative

$$\begin{array}{ccc}
\text{Hom}_{\mathcal{D}^{op}}(M, F(N)) & \longrightarrow & \text{Hom}_{\mathcal{C}}(N^{\vee}, F'(M^{\vee})) \\
\downarrow & & \downarrow \gamma \\
\text{Hom}_{\mathcal{D}}(F(N)^{\vee}, M^{\vee}) & \longrightarrow & \text{Hom}_{\mathcal{D}}(G'(N^{\vee}), M^{\vee})
\end{array} \tag{1.6}$$

for $M \in \text{Per}(\mathcal{D})$ and $N \in \text{Per}(\mathcal{C})$. In this diagram the upper horizontal arrow is an isomorphism of part (i), the lower horizontal arrow is induced by (1.5), and γ is the adjunction isomorphism of Lemma 1.1.

Proof. (i) By (1.2), we have

$$\text{Hom}(M, F(N)) \simeq F(N) \otimes_{\mathcal{D}}^{\mathbb{L}} M^{\vee} \simeq N \otimes_{\mathcal{C}}^{\mathbb{L}} K \otimes_{\mathcal{D}}^{\mathbb{L}} M^{\vee}.$$

Reversing the order of factors we can rewrite this as

$$M^{\vee} \otimes_{\mathcal{D}^{op}}^{\mathbb{L}} K' \otimes_{\mathcal{C}^{op}}^{\mathbb{L}} N \simeq F'(M^{\vee}) \otimes_{\mathcal{C}^{op}}^{\mathbb{L}} N \simeq \text{Hom}(N^{\vee}, F'(M^{\vee})).$$

(ii) First, taking cofibrant resolutions for K' and $(K^T)'$, we can realize the map γ (with M^{\vee} replaced by any \mathcal{D} -module L) on the chain level as the following composition:

$$\begin{aligned}
& \text{Hom}(N^{\vee}, L \otimes_{\mathcal{D}^{op}} K') \rightarrow \text{Hom}(N^{\vee} \otimes_{\mathcal{C}^{op}} (K^T)', L \otimes_{\mathcal{D}^{op}} K' \otimes_{\mathcal{C}^{op}} (K^T)') \rightarrow \\
& \text{Hom}(N^{\vee} \otimes_{\mathcal{C}^{op}} (K^T)', L),
\end{aligned} \tag{1.7}$$

where the second arrow is induced by the map

$$K' \otimes_{\mathcal{C}^{op}} (K^T)' \rightarrow (K^T)'^T \otimes_{\mathcal{C}^{op}} (K^T)' \rightarrow \Delta_{\mathcal{D}^{op}}$$

in $\text{Per}_{dg}(\mathcal{D} \otimes \mathcal{D}^{op})$. Composing (1.7) with a quasi-isomorphism of the form $? \otimes_{\mathcal{C}^{op}} N \rightarrow \text{Hom}(N^{\vee}, ?)$ obtained from (1.2), we get a quasi-isomorphism

$$L \otimes_{\mathcal{D}^{op}} K' \otimes_{\mathcal{C}^{op}} N \rightarrow \text{Hom}(N^{\vee} \otimes_{\mathcal{C}^{op}} (K^T)', L).$$

Note that $K' \otimes_{\mathcal{C}^{op}} N \simeq N \otimes_{\mathcal{C}} K$. Thus, for $L = h_{D^{\vee}}$, where $D^{\vee} \in \mathcal{D}^{op}$, this gives a quasi-isomorphism of dg-modules

$$(N \otimes_{\mathcal{C}} K)(D^{\vee}) \rightarrow (N^{\vee} \otimes_{\mathcal{C}^{op}} (K^T)')^{\vee}(D^{\vee}),$$

hence we get a quasi-isomorphism $F(N) \rightarrow G'(N^{\vee})^{\vee}$, from which (1.5) is obtained by duality.

Now using the isomorphisms (1.2) we can replace (1.6) with the following diagram in which we replaced M^{\vee} by L and inserted an additional diagonal arrow:

$$\begin{array}{ccc}
N \otimes_{\mathcal{C}} K \otimes_{\mathcal{D}} L & \longrightarrow & L \otimes_{\mathcal{D}^{op}} K' \otimes_{\mathcal{C}^{op}} N \\
\downarrow & \nearrow & \downarrow \\
L \otimes_{\mathcal{D}^{op}} (N \otimes_{\mathcal{C}} K) & \longrightarrow & L \otimes_{\mathcal{D}^{op}} G'(N^{\vee})^{\vee}
\end{array} \tag{1.8}$$

It remains to observe that in this diagram the upper left triangle is commutative for trivial reasons, and the lower right triangle is commutative by the above construction of the map $F(N) \rightarrow G'(N^{\vee})^{\vee}$. \square

Let us consider the external tensor product $K' \square K^T \in \text{Per}(\mathcal{D} \otimes \mathcal{D}^{op} \otimes \mathcal{C}^{op} \otimes \mathcal{C})$. Then we have a natural isomorphism

$$(K' \square K^T) \otimes_{\mathcal{C}^{op} \otimes \mathcal{C}}^{\mathbb{L}} \Delta_{\mathcal{C}^{op}} \xrightarrow{\sim} (K^T \otimes_{\mathcal{C}} K) \circ \sigma$$

in $\text{Per}(\mathcal{D}^{op} \otimes \mathcal{D})$ (cf. [22, (1.8)]). Combining it with the permutation of (1.3) we get a map of kernels

$$(K' \square K^T) \otimes_{\mathcal{C}^{op} \otimes \mathcal{C}}^{\mathbb{L}} \Delta_{\mathcal{C}^{op}} \rightarrow \Delta_{\mathcal{D}^{op}} \tag{1.9}$$

in $\text{Per}(\mathcal{D} \otimes \mathcal{D}^{op})$. Note also that the functor $F' \square G : \text{Per}(\mathcal{D}^{op} \otimes \mathcal{D}) \rightarrow \text{Per}(\mathcal{C}^{op} \otimes \mathcal{C})$ given by the kernel $K' \square K^T$, viewed as an operation on functors, acts by $\Phi \mapsto G \circ \Phi \circ F$. In particular, $(F' \square G)(\Delta_{\mathcal{D}})$ is the kernel giving $G \circ F$.

1.3. Connection with traces. Recall that we have the natural trace functors

$$\text{Tr}_{\mathcal{C}} : \text{Per}(\mathcal{C}^{op} \otimes \mathcal{C}) \rightarrow \text{Per}(k)$$

$$\text{Tr}_{\mathcal{C}}^{dg} : \text{Per}_{dg}(\mathcal{C}^{op} \otimes \mathcal{C}) \rightarrow \text{Per}_{dg}(k)$$

given by the tensor product on the right with the diagonal $\mathcal{C}^{op} \otimes \mathcal{C}$ -module $\Delta_{\mathcal{C}^{op}}$ (see [29, Sec. 5.2.3] and [22, Sec. 1.1]). In other words, $\text{Tr}_{\mathcal{C}}$ is the functor of Hochschild homology with coefficients in a bimodule. By abuse of notation, for a tensor endofunctor $F : \text{Per}(\mathcal{C}) \rightarrow \text{Per}(\mathcal{C})$ given by a kernel $K \in \text{Per}(\mathcal{C}^{op} \otimes \mathcal{C})$ we will write $\text{Tr}_{\mathcal{C}}(F)$ instead of $\text{Tr}_{\mathcal{C}}(K)$. Recall that for any tensor functors $F_1 : \text{Per}(\mathcal{C}) \rightarrow \text{Per}(\mathcal{D})$ and $F_2 : \text{Per}(\mathcal{D}) \rightarrow \text{Per}(\mathcal{C})$ we have a canonical isomorphism

$$\text{Tr}_{\mathcal{D}}(F_1 \circ F_2) \simeq \text{Tr}_{\mathcal{C}}(F_2 \circ F_1) \quad (1.10)$$

(see [22, Lem. 1.1.3]).

Returning to the situation of Section 1.2 we observe that the map of kernels (1.9) induces a natural transformation

$$t(F, G) : \text{Tr}_{\mathcal{C}} \circ (F' \square G) \rightarrow \text{Tr}_{\mathcal{D}}. \quad (1.11)$$

By Lemma 1.1, the above construction is also applicable to the adjoint pair of functors (G', F') , so it gives a map

$$t(G', F') : \text{Tr}_{\mathcal{C}^{op}} \circ (G \square F') \rightarrow \text{Tr}_{\mathcal{D}^{op}}.$$

It is easy to see that the following diagram of functors $\text{Per}(\mathcal{D} \otimes \mathcal{D}^{op}) \rightarrow \text{Per}(k)$ is commutative

$$\begin{array}{ccc} \text{Tr}_{\mathcal{C}} \circ (F' \square G) \circ \sigma_{\mathcal{D}} & \xrightarrow{t(F, G) \circ \sigma_{\mathcal{D}}} & \text{Tr}_{\mathcal{D}} \circ \sigma_{\mathcal{D}} \\ \sim \downarrow & & \sim \downarrow \\ \text{Tr}_{\mathcal{C}^{op}} \circ (G \square F') & \xrightarrow{t(G', F')} & \text{Tr}_{\mathcal{D}^{op}} \end{array} \quad (1.12)$$

where $\sigma_{\mathcal{D}} : \text{Per}(\mathcal{D} \otimes \mathcal{D}^{op}) \rightarrow \text{Per}(\mathcal{D}^{op} \otimes \mathcal{D})$ is the permutation equivalence, the isomorphism in the left column is induced by the isomorphism

$$(F' \square G) \circ \sigma_{\mathcal{D}} \simeq \sigma_{\mathcal{C}} \circ (G \square F')$$

and by the isomorphism $\text{Tr}_{\mathcal{C}} \circ \sigma_{\mathcal{C}} \simeq \text{Tr}_{\mathcal{C}^{op}}$.

Note that we have a natural isomorphism

$$(F' \square G)(\Delta_{\mathcal{D}}) \simeq \Delta_{\mathcal{D}} \otimes_{\mathcal{D}^{op} \otimes \mathcal{D}} (K' \square K^T) \simeq K \otimes_{\mathcal{D}} K^T.$$

Hence, the canonical adjunction map (1.4) can be viewed as a map

$$\gamma_{F, G} : \Delta_{\mathcal{C}} \rightarrow (F' \square G)(\Delta_{\mathcal{D}}). \quad (1.13)$$

Using isomorphism of Lemma 1.2(i) for the map

$$\gamma_{G', F'} : \Delta_{\mathcal{C}^{op}} \rightarrow (G \square F')(\Delta_{\mathcal{D}^{op}})$$

we obtain a canonical morphism

$$\Delta_{\mathcal{D}^{op}}^{\vee} \rightarrow (G' \square F)(\Delta_{\mathcal{C}^{op}}^{\vee}). \quad (1.14)$$

Note also that for $M \in \text{Per}(\mathcal{C}^{op} \otimes \mathcal{C})$ we have a canonical isomorphism

$$\text{Hom}(\Delta_{\mathcal{C}^{op}}^{\vee}, M) \simeq M \otimes_{\mathcal{C}^{op} \otimes \mathcal{C}}^{\mathbb{L}} \Delta_{\mathcal{C}^{op}} = \text{Tr}_{\mathcal{C}}(M). \quad (1.15)$$

Lemma 1.3. (i) For $\Phi \in \text{Per}(\mathcal{D}^{op} \otimes \mathcal{D})$ the natural transformation

$$t(F, G)(\Phi) : \text{Tr}_{\mathcal{C}}((F' \square G)(\Phi)) \rightarrow \text{Tr}_{\mathcal{D}}(\Phi)$$

is given by the composition

$$\text{Tr}_{\mathcal{C}}((F' \square G)(\Phi)) \simeq \text{Tr}_{\mathcal{C}}(G \circ \Phi \circ F) \simeq \text{Tr}_{\mathcal{D}}(F \circ G \circ \Phi) \rightarrow \text{Tr}_{\mathcal{D}}(\Phi),$$

where the second isomorphism uses (1.10).

(ii) Via the identification (1.15), $t(F, G)(\Phi)$ can be identified with the composition

$$\text{Hom}(\Delta_{\mathcal{C}^{op}}^{\vee}, (F' \square G)(\Phi)) \xrightarrow{\sim} \text{Hom}((G' \square F)(\Delta_{\mathcal{C}^{op}}^{\vee}), \Phi) \rightarrow \text{Hom}(\Delta_{\mathcal{D}^{op}}^{\vee}, \Phi), \quad (1.16)$$

where the first arrow is given by adjunction of the pair $((G' \square F), (F' \square G))$, while the second is induced by (1.14).

Proof. (i) Consider the object

$$L = K \boxtimes \Phi \boxtimes K^T \in \text{Per}(\mathcal{C}^{op} \otimes \mathcal{D} \otimes \mathcal{D}^{op} \otimes \mathcal{D} \otimes \mathcal{D}^{op} \otimes \mathcal{C}).$$

We have

$$\text{Tr}_{\mathcal{C}}(G \circ \Phi \circ F) = \text{Tr}_{\mathcal{C}}(K \otimes_{\mathcal{D}} \Phi \otimes_{\mathcal{D}} K^T) = \text{Tr}_{\mathcal{C}}((\Delta_{\mathcal{D}}^{23} \boxtimes \Delta_{\mathcal{D}}^{45}) \otimes_{\mathcal{D}^{(4)}} L), \quad (1.17)$$

where $\mathcal{D}^{(4)} = \mathcal{D}^{op} \otimes \mathcal{D} \otimes \mathcal{D}^{op} \otimes \mathcal{D}$. On the other hand,

$$\text{Tr}_{\mathcal{D}}(F \circ G \circ \Phi) = \text{Tr}_{\mathcal{D}}(\Delta_{\mathcal{D}}^{45} \otimes_{\mathcal{D}^{op} \otimes \mathcal{D}} L \otimes_{\mathcal{C}^{op} \otimes \mathcal{C}} \Delta_{\mathcal{C}^{op}}^{16}). \quad (1.18)$$

Now the isomorphism between (1.18) and (1.17) is obtained by identifying both with

$$(\Delta_{\mathcal{D}}^{23} \boxtimes \Delta_{\mathcal{D}}^{45}) \otimes_{\mathcal{D}^{(4)}} L \otimes_{\mathcal{C}^{op} \otimes \mathcal{C}} \Delta_{\mathcal{C}^{op}}^{16}.$$

The map $\text{Tr}_{\mathcal{D}}(F \circ G \circ \Phi) \rightarrow \text{Tr}_{\mathcal{D}}(\Phi)$ is induced by the adjunction morphism $F \circ G \rightarrow \Delta_{\mathcal{D}}$, i.e., in terms of the above isomorphism by the map

$$L \otimes_{\mathcal{C}^{op} \otimes \mathcal{C}} \Delta_{\mathcal{C}^{op}}^{16} \rightarrow (\Phi \boxtimes \Delta_{\mathcal{D}}) \circ \sigma_{4321}$$

induced by (1.3), where σ_{4321} is the cyclic permutation of factors in the tensor product. Now the required compatibility follows from the commutativity of the diagram

$$\begin{array}{ccc} \Phi \otimes_{\mathcal{D}^{op} \otimes \mathcal{D}} (K' \square K^T) \otimes_{\mathcal{C}^{op} \otimes \mathcal{C}} \Delta_{\mathcal{C}^{op}} & \xrightarrow{\quad} & \Phi \otimes_{\mathcal{D}^{op} \otimes \mathcal{D}} \Delta_{\mathcal{D}^{op}} \\ \sim \downarrow & & \sim \downarrow \\ (\Delta_{\mathcal{D}}^{23} \boxtimes \Delta_{\mathcal{D}}^{45}) \otimes_{\mathcal{D}^{(4)}} L \otimes_{\mathcal{C}^{op} \otimes \mathcal{C}} \Delta_{\mathcal{C}^{op}}^{16} & \xrightarrow{\quad} & (\Delta_{\mathcal{D}}^{23} \boxtimes \Delta_{\mathcal{D}}^{45}) \otimes_{\mathcal{D}^{(4)}} [(\Phi \boxtimes \Delta_{\mathcal{D}}) \circ \sigma_{4321}] \end{array}$$

in which both horizontal arrows are induced by (1.3).

(ii) Applying the commutative diagram of Lemma 1.2(ii) to the adjoint pair $((G' \square F), (F' \square G))$ and the pair of objects $(\Delta_{\mathcal{C}^{op}}, \Delta_{\mathcal{D}^{op}})$ we obtain that the dual map to (1.14),

$$(F \square G')(\Delta_{\mathcal{C}^{op}}) \simeq ((G' \square F)(\Delta_{\mathcal{C}^{op}}^{\vee}))^{\vee} \rightarrow \Delta_{\mathcal{D}^{op}}$$

corresponds by adjointness to $\gamma_{G', F'}$. Hence, this map is given by the natural morphism

$$(F \square G')(\Delta_{\mathcal{C}^{op}}) \simeq (K' \square K^T) \otimes_{\mathcal{C}^{op} \otimes \mathcal{C}} \Delta_{\mathcal{C}^{op}} \rightarrow \Delta_{\mathcal{D}^{op}}$$

(see (1.9)). Now rewriting the composition (1.16) using the isomorphisms of the form (1.2) we obtain the result. \square

2. GENERALIZED FUNCTORIALITY FOR HOCHSCHILD HOMOLOGY WITH COEFFICIENTS

2.1. Generalized functoriality. Let (\mathcal{C}_1, F_1) and (\mathcal{C}_2, F_2) be a pair of dg-categories equipped with endofunctors F_i of $\text{Per}_{dg}(\mathcal{C}_i)$.

Definition. (i) A *dg-functor*

$$(\Phi, \phi) : (\text{Per}_{dg}(\mathcal{C}_1), F_1) \rightarrow (\text{Per}_{dg}(\mathcal{C}_2), F_2) \quad (2.1)$$

consists of a dg-functor $\Phi : \text{Per}_{dg}(\mathcal{C}_1) \rightarrow \text{Per}_{dg}(\mathcal{C}_2)$ together with a (closed) morphism of dg-functors

$$\phi : \Phi \circ F_1 \rightarrow F_2 \circ \Phi.$$

(ii) If (\mathcal{C}_3, F_3) is another dg-category with an endofunctor of $\text{Per}_{dg}(\mathcal{C}_3)$ and

$$(\Phi', \phi') : (\text{Per}_{dg}(\mathcal{C}_2), F_2) \rightarrow (\text{Per}_{dg}(\mathcal{C}_3), F_3)$$

is a dg-functor, then we have an induced morphism

$$\phi' \circ \phi : \Phi' \circ \Phi \circ F_1 \rightarrow \Phi' \circ F_2 \circ \Phi \rightarrow F_3 \circ \Phi' \circ \Phi$$

and we define the composition of these dg-functors by

$$(\Phi', \phi') \circ (\Phi, \phi) = (\Phi' \circ \Phi, \phi' \circ \phi) : (\text{Per}_{dg}(\mathcal{C}_1), F_1) \rightarrow (\text{Per}_{dg}(\mathcal{C}_3), F_3).$$

Given a dg-functor (2.1) we define the induced morphism

$$(\Phi, \phi)_* : \text{Tr}_{\mathcal{C}_1}(F_1) \rightarrow \text{Tr}_{\mathcal{C}_2}(F_2) \quad (2.2)$$

as the composition

$$\text{Tr}(F_1) \rightarrow \text{Tr}(\Psi \circ \Phi \circ F_1) \rightarrow \text{Tr}(\Phi \circ F_1 \circ \Psi) \rightarrow \text{Tr}(F_2 \circ \Phi \circ \Psi) \rightarrow \text{Tr}(F_2),$$

where Ψ is the right adjoint functor to Φ .

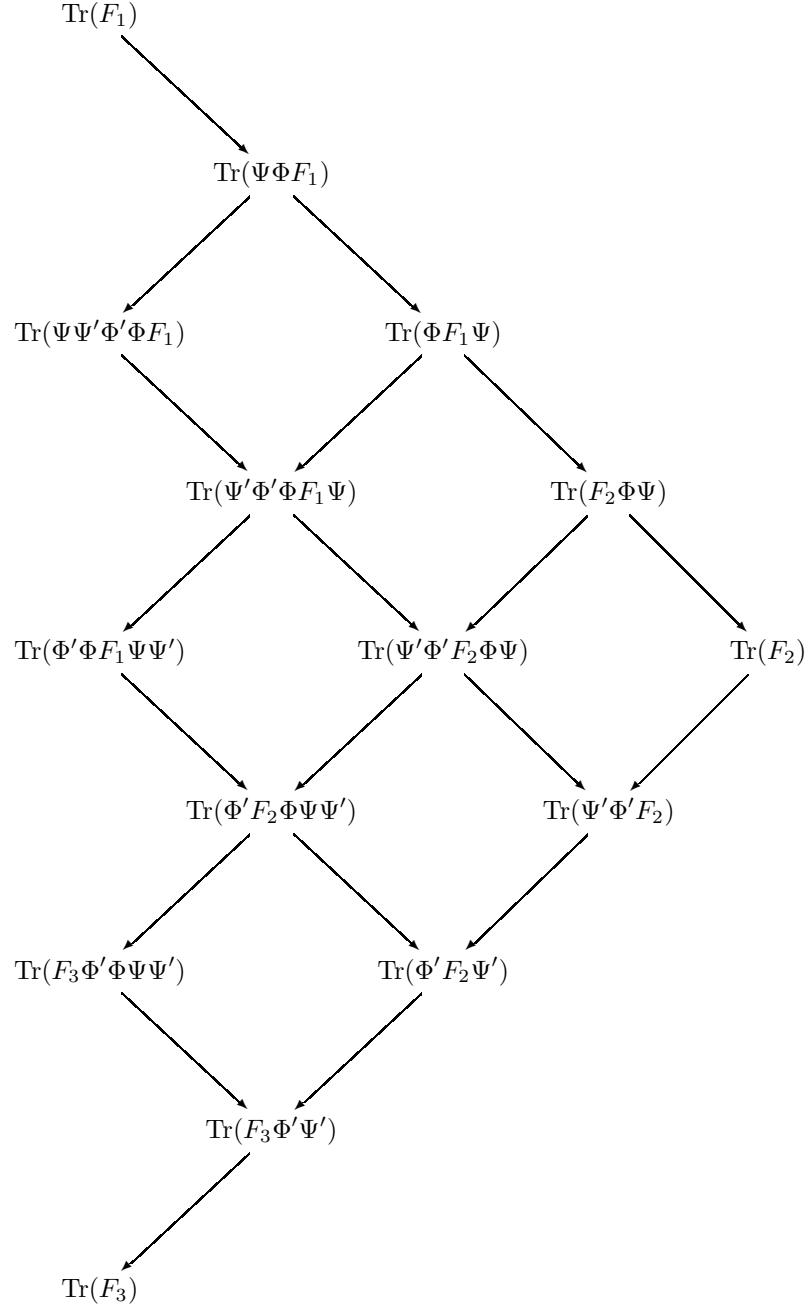
The above construction is compatible with compositions.

Lemma 2.1. *In the situation of Definition (2.1)(ii) one has*

$$(\Phi' \circ \Phi, \phi' \circ \phi)_* = (\Phi', \phi')_* \circ (\Phi, \phi)_*$$

in $\text{Hom}_k(\text{Tr}_{\mathcal{C}_1}(F_1), \text{Tr}_{\mathcal{C}_3}(F_3))$.

Proof. This follows from the commutativity of the diagram (where we omitted the \circ symbols)



Namely, $(\Phi' \circ \Phi, \phi' \circ \phi)_*$ is equal to the composition of the arrows going from $\text{Tr}(F_1)$ to $\text{Tr}(F_3)$ along the left edge of the diagram, while $(\Phi', \phi')_* \circ (\Phi, \phi)_*$ is the composition of the arrows along the right edge of the diagram. \square

Remark. In the case when F_1 and F_2 are the identity functors and $\phi = \text{id}$, the associated map (2.2) is the functoriality map $\Phi_* : HH_*(\mathcal{C}_1) \rightarrow HH_*(\mathcal{C}_2)$ defined in [22, Sec. 1.2]. The equivalence of this

definition of F_* with the standard definition of the functoriality using Hochschild complexes is checked in [24, Appendix].

2.2. Generalized boundary-bulk map. Given a tensor functor $F : \text{Per}(\mathcal{C}) \rightarrow \text{Per}(\mathcal{C})$ (given by a kernel $K \in \text{Per}(\mathcal{C}^{op} \otimes \mathcal{C})$) and an object $A \in \text{Per}(\mathcal{C})$, we can construct a natural map

$$\tau_F^A : \text{Hom}(A, F(A)) \rightarrow \text{Tr}_{\mathcal{C}}(F)$$

which in the case $F = \text{Id}$ becomes the boundary-bulk map defined in [22, Prop. 1.2.4]. Namely, this map is defined by applying the functor $\text{Tr}_{\mathcal{C}}$ to the natural morphism

$$c_F^A : A^\vee \boxtimes F(A) \rightarrow F \quad (2.3)$$

in $D(\mathcal{C}^{op} \otimes \mathcal{C})$ (see [22, Eq. (1.25)]). It is useful to note that the map c_F^A is obtained by applying $\text{Id}_{\mathcal{C}} \square F$ to the morphism

$$c_{\text{Id}}^A : A^\vee \boxtimes A \rightarrow \Delta_{\mathcal{C}}, \quad (2.4)$$

or equivalently, by composing the corresponding functors with F on the left. (see the proof of Lemma 1.2.5(ii) in [22]).

Note that we can view an object A as a dg-functor $\iota_A : \text{Per}_{dg}(k) \rightarrow \text{Per}_{dg}(\mathcal{C})$ sending k to A . Then a map $\alpha : A \rightarrow F(A)$ is the same as a morphism of dg-functors $\alpha : \iota_A \rightarrow F \circ \iota_A$. Thus, we have a dg-functor

$$(\iota_A, \alpha) : (\text{Per}_{dg}(k), \text{Id}) \rightarrow (\text{Per}_{dg}(\mathcal{C}), F)$$

and the construction of 2.1 gives an induced map

$$(\iota_A, \alpha)_* : k \rightarrow \text{Tr}_{\mathcal{C}}(F).$$

Lemma 2.2. *One has*

$$\tau_F^A(\alpha) = (\iota_A, \alpha)_*(1).$$

Proof. The right quasi-adjoint functor Ψ to $\Phi = \iota_A$ is given by the kernel A^\vee , viewed as a $\mathcal{C} - k$ -bimodule, so $\Psi \circ \Phi$ corresponds to the vector space $\text{Hom}(A, A)$ while $\Phi \circ \Psi$ corresponds to the kernel $A^\vee \boxtimes A$. Thus, by definition, $(\iota_A, \alpha)_*$ is the following composition

$$k \xrightarrow{\cdot \text{id}_A} \text{Hom}(A, A) \xrightarrow{\sim} \text{Tr}_{\mathcal{C}}(\Phi \circ \Psi) \rightarrow \text{Tr}_{\mathcal{C}}(F \circ \Phi \circ \Psi) \rightarrow \text{Tr}_{\mathcal{C}}(F).$$

Note that the last two arrows are obtained by applying $\text{Tr}_{\mathcal{C}}$ to the morphisms

$$A^\vee \boxtimes A \xrightarrow{\text{id} \otimes \alpha} A^\vee \boxtimes F(A) \quad \text{and}$$

$$A^\vee \boxtimes F(A) \simeq F \circ (A^\vee \boxtimes A) \rightarrow F \circ \Delta_{\mathcal{C}}.$$

Now the asserted equality follows from the fact the last composition is equal to (2.3). \square

Recall that we have an isomorphism

$$\text{Hom}(A, F(A)) \simeq \text{Hom}(A^\vee, F'(A^\vee)), \quad (2.5)$$

where F' is given by the kernel $K' = K \circ \sigma$ (see Lemma 1.2(i)).

Lemma 2.3. *Let $\alpha' : A^\vee \rightarrow F'(A^\vee)$ be a morphism associated with a morphism $\alpha : A \rightarrow F(A)$ under (2.5). Then*

$$\tau_F^A(\alpha) = \tau_{F'}^{A^\vee}(\alpha')$$

in $\text{Tr}_{\mathcal{C}}(F) \simeq \text{Tr}_{\mathcal{C}^{op}}(F')$.

Proof. The isomorphism (2.5) can be obtained from the isomorphisms

$$\mathrm{Hom}(A, F(A)) \simeq \mathrm{Tr}_{\mathcal{C}}((\mathrm{Id}_{\mathcal{C}^{op}} \square F)(A^\vee \boxtimes A)) \simeq \mathrm{Tr}(F \circ X) \quad \text{and}$$

$$\mathrm{Hom}(A^\vee, F'(A^\vee)) \simeq \mathrm{Tr}_{\mathcal{C}^{op}}((\mathrm{Id}_{\mathcal{C}} \square F')(A \boxtimes A^\vee)) \simeq \mathrm{Tr}_{\mathcal{C}}((F' \square \mathrm{Id}_{\mathcal{C}^{op}})(A^\vee \boxtimes A)) \simeq \mathrm{Tr}(X \circ F),$$

where $X = A^\vee \boxtimes A$, together with an isomorphism $\mathrm{Tr}(X \circ F) \simeq \mathrm{Tr}(F \circ X)$. Thus, it remains to check that the maps $F \circ X \rightarrow F$ and $X \circ F \rightarrow F$ induced by the canonical morphism $X \rightarrow \Delta_{\mathcal{C}}$, give rise to the same maps after applying Tr . This boils down to the fact that the isomorphism

$$\mathrm{Tr}(F) \simeq \mathrm{Tr}(\Delta \circ F) \simeq \mathrm{Tr}(F \circ \Delta) \simeq \mathrm{Tr}(F)$$

is equal to the identity. \square

As in the case of the generalized HRR-theorem of [22, Lem. 1.3.2], the fixed point formula will be deduced from the following compatibility of the boundary-bulk maps with dg-functors.

Lemma 2.4. *Let (\mathcal{C}_1, F_1) and (\mathcal{C}_2, F_2) be a pair of dg-categories equipped with endofunctors. Suppose we have a dg-functor $(\Phi, \phi) : (\mathrm{Per}_{dg}(\mathcal{C}_1), F_1) \rightarrow (\mathrm{Per}_{dg}(\mathcal{C}_2), F_2)$. For an object $A \in \mathrm{Per}(\mathcal{C}_1)$ let us consider the composed map*

$$(\Phi, \phi)_*^A : \mathrm{Hom}(A, F_1 A) \xrightarrow{\Phi} \mathrm{Hom}(\Phi(A), \Phi(F_1 A)) \rightarrow \mathrm{Hom}(\Phi(A), F_2 \Phi(A)),$$

where the second arrow is induced by ϕ . Then the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Hom}(A, F_1 A) & \xrightarrow{(\Phi, \phi)_*^A} & \mathrm{Hom}(\Phi(A), F_2 \Phi(A)) \\ \tau_{F_1}^A \downarrow & & \downarrow \tau_{F_2}^{\Phi(A)} \\ \mathrm{Tr}(F_1) & \xrightarrow{(\Phi, \phi)_*} & \mathrm{Tr}(F_2) \end{array}$$

where the bottom horizontal arrow is the map (2.2).

Proof. The idea is to use the compatibility of the generalized functoriality with compositions (see Lemma 2.1). Namely, as we have seen above, an element $\alpha \in \mathrm{Hom}(A, F_1 A)$ gives rise to a dg-functor

$$(\iota_A, \alpha) : (\mathrm{Per}_{dg}(k), \mathrm{Id}) \rightarrow (\mathrm{Per}_{dg}(\mathcal{C}_1), F_1)$$

such that

$$\tau_{F_1}^A(\alpha) = (\iota_A, \alpha)_*(1)$$

(see Lemma 2.2). Let $\alpha' = (\Phi, \phi)_*^A(\alpha)$. It follows directly from the definitions that the dg-functor

$$(\iota_{\Phi(A)}, \alpha')(\mathrm{Per}_{dg}(k), \mathrm{Id}) \rightarrow (\mathrm{Per}_{dg}(\mathcal{C}_2), F_2)$$

is isomorphic to the composition $(\Phi, \phi) \circ (\iota_A, \alpha)$. Hence, by Lemma 2.1, we obtain

$$(\iota_{\Phi(A)}, \alpha')_*(1) = (\Phi, \phi)_*(\iota_A, \alpha)_*(1) = (\Phi, \phi)_*(\tau_{F_1}^A(\alpha)).$$

It remains to apply Lemma 2.2 to see that the left-hand side is equal to $\tau_{F_2}^{\Phi(A)}(\alpha')$. \square

2.3. Canonical pairings. We can also use the functoriality to define a canonical pairing

$$\langle \cdot, \cdot \rangle_{F,G} : \text{Tr}(F) \otimes \text{Tr}(G) \rightarrow k, \quad (2.6)$$

where $F : \text{Per}(\mathcal{C}) \rightarrow \text{Per}(\mathcal{C})$ is given by a kernel K , and $G : \text{Per}(\mathcal{C}) \rightarrow \text{Per}(\mathcal{C})$ is the right adjoint functor to F associated with the kernel K^T (see Sec. 1). Recall that $F' : \text{Per}(\mathcal{C}^{op}) \rightarrow \text{Per}(\mathcal{C}^{op})$ denotes the functor associated with the kernel $K' = K \circ \sigma \in \text{Per}(\mathcal{C} \otimes \mathcal{C}^{op})$. We can apply the functoriality to the functor

$$\text{Tr}_{\mathcal{C}}^{dg} : \text{Per}_{dg}(\mathcal{C}^{op} \otimes \mathcal{C}) \rightarrow \text{Per}_{dg}(k),$$

the identity endofunctor of $\text{Per}_{dg}(k)$, and the endofunctor $F' \square G$ of $\text{Per}_{dg}(\mathcal{C}^{op} \otimes \mathcal{C})$. Note that we have a morphism

$$t(F, G) : \text{Tr}_{\mathcal{C}}^{dg} \circ (F' \square G) \rightarrow \text{Tr}_{\mathcal{C}}^{dg}$$

(this is (1.11) in the case $\mathcal{D} = \mathcal{C}$). Thus, we can extend $\text{Tr}_{\mathcal{C}}^{dg}$ to a dg-functor

$$(\text{Tr}_{\mathcal{C}}^{dg}, t(F, G)) : (\text{Per}_{dg}(\mathcal{C}^{op} \otimes \mathcal{C}), F' \square G) \rightarrow (\text{Per}_{dg}(k), \text{Id}). \quad (2.7)$$

Hence, by the construction of Section 2.1, we obtain a map

$$(\text{Tr}_{\mathcal{C}}^{dg}, t(F, G))_* : \text{Tr}_{\mathcal{C}^{op} \otimes \mathcal{C}}(F' \square G) \rightarrow \text{Tr}_k(\Delta_k) = k.$$

Using the isomorphisms

$$\text{Tr}_{\mathcal{C}^{op} \otimes \mathcal{C}}(F' \square G) \simeq \text{Tr}_{\mathcal{C}^{op}}(F') \otimes \text{Tr}_{\mathcal{C}}(G)$$

and $\text{Tr}_{\mathcal{C}^{op}}(F') \simeq \text{Tr}_{\mathcal{C}}(F)$ we obtain the pairing (2.6).

Applying the above construction to the quasi-adjoint pair (G', F') (see Lemma 1.1) we similarly get a pairing

$$\langle \cdot, \cdot \rangle_{G', F'} : \text{Tr}(G') \otimes \text{Tr}(F') \rightarrow k.$$

Lemma 2.5. *For $x \in \text{Tr}(F) \simeq \text{Tr}(F')$ and $y \in \text{Tr}(G) \simeq \text{Tr}(G')$ one has*

$$\langle y, x \rangle_{G', F'} = \pm \langle x, y \rangle_{F, G},$$

where the sign is given by the Koszul rule.

Proof. This follows easily from the commutativity of the diagram (1.12). \square

Recall that we have a canonical map

$$\gamma_{F,G} : \Delta_{\mathcal{C}} \rightarrow (F' \square G)(\Delta_{\mathcal{C}})$$

(see (1.13))

Lemma 2.6. (i) *The element*

$$\tau_{F,G} := \tau_{F' \square G}^{\Delta_{\mathcal{C}}}(\gamma_{F,G}) \in \text{Tr}(F') \otimes \text{Tr}(G)$$

satisfies

$$(\langle \cdot, \cdot \rangle_{G', F'} \otimes \text{id})(x \otimes \tau_{F,G}) = x, \quad (2.8)$$

for any $x \in \text{Tr}(G') \simeq \text{Tr}(G)$.

(ii) *The pairing (2.6) is perfect, and $\tau_{F,G} \in \text{Tr}(F') \otimes \text{Tr}(G) \simeq \text{Tr}(F) \otimes \text{Tr}(G)$ is the corresponding Casimir element.*

Proof. (i) By Lemma 2.2, we can view $\tau_{F' \square G}^{\Delta_{\mathcal{C}}}(\gamma_{F,G})$ as the generalized functoriality map associated with the dg-functor

$$(\delta, \gamma_{F,G}) : (\text{Per}_{dg}(k), \text{Id}) \rightarrow (\text{Per}_{dg}(\mathcal{C}^{op} \otimes \mathcal{C}), F' \square G)$$

where $\delta : \text{Per}_{dg}(k) \rightarrow \text{Per}_{dg}(\mathcal{C}^{op} \otimes \mathcal{C})$ sends k to $\Delta_{\mathcal{C}}$. It is easy to see that the composition

$$\text{Per}_{dg}(\mathcal{C}) \xrightarrow{\text{Id}_{\mathcal{C}} \square \delta} \text{Per}_{dg}(\mathcal{C} \otimes \mathcal{C}^{op} \otimes \mathcal{C}) \xrightarrow{\text{Tr}_{\mathcal{C}^{op}} \square \text{Id}_{\mathcal{C}}} \text{Per}_{dg}(\mathcal{C}) \quad (2.9)$$

is isomorphic to the identity. Indeed, for any kernel $K \in \text{Per}_{dg}(\mathcal{C}^{op} \otimes \mathcal{C})$ and $M \in \text{Per}_{dg}(\mathcal{C})$ we have

$$(\text{Tr}_{\mathcal{C}^{op}} \square \text{Id}_{\mathcal{C}})(M \boxtimes K) \simeq M \otimes_{\mathcal{C}} K.$$

Thus, for $K = \Delta_{\mathcal{C}}$ we get

$$(\text{Tr}_{\mathcal{C}^{op}} \square \text{Id}_{\mathcal{C}})(M \boxtimes \Delta_{\mathcal{C}}) \simeq M \otimes_{\mathcal{C}} \Delta_{\mathcal{C}} \simeq M.$$

Next, we observe that both arrows in (2.9) extend to dg-functors between dg-categories with endofunctors. Namely, we have dg-functors

$$(\text{Id}_{\mathcal{C}} \square \delta, \text{id} \square \gamma_{F,G}) : (\text{Per}_{dg}(\mathcal{C}), G) \rightarrow (\text{Per}_{dg}(\mathcal{C} \otimes \mathcal{C}^{op} \otimes \mathcal{C}), G \square F' \square G)$$

and

$$(\text{Tr}_{\mathcal{C}^{op}}^{dg} \square \text{Id}_{\mathcal{C}}, t(G', F') \square \text{id}) : (\text{Per}_{dg}(\mathcal{C} \otimes \mathcal{C}^{op} \otimes \mathcal{C}), G \square F' \square G) \rightarrow (\text{Per}_{dg}(\mathcal{C}), G)$$

(where we use (2.7) for the pair (G', F')). We claim that the composition of these dg-functors is

$$(\text{Tr}_{\mathcal{C}^{op}}^{dg} \square \text{Id}_{\mathcal{C}}, t(G', F') \square \text{id}) \circ (\text{Id}_{\mathcal{C}} \square \delta, \text{id} \square \gamma_{F,G}) \simeq (\text{Id}_{\mathcal{C}}, \text{id}). \quad (2.10)$$

For this we have to check that the composition

$$\begin{aligned} G(M) &\simeq (\text{Tr}_{\mathcal{C}^{op}} \square \text{Id}_{\mathcal{C}})(G(M) \boxtimes \Delta_{\mathcal{C}}) \xrightarrow{\text{id} \otimes \gamma_{F,G}} (\text{Tr}_{\mathcal{C}^{op}} \square \text{Id}_{\mathcal{C}}) \circ (G \square F' \square G)(M \boxtimes \Delta_{\mathcal{C}}) \\ &\xrightarrow{t(G', F') \otimes \text{id}} (\text{Tr}_{\mathcal{C}^{op}} \square G)(M \boxtimes \Delta_{\mathcal{C}}) \simeq G \circ (\text{Tr}_{\mathcal{C}^{op}} \square \text{Id})(M \boxtimes \Delta_{\mathcal{C}}) \simeq G(M) \end{aligned}$$

is the identity. The corresponding composition of maps between the kernels can be rewritten as

$$K^T \simeq \text{Tr}^{23}(K^T \boxtimes \Delta_{\mathcal{C}}) \longrightarrow \text{Tr}^{23}(K^T \boxtimes (K \otimes_{\mathcal{C}} K^T)) \simeq \text{Tr}^{23}(K^T \boxtimes K) \otimes_{\mathcal{C}} K^T \longrightarrow \Delta_{\mathcal{C}} \otimes_{\mathcal{C}} K^T \simeq K^T,$$

where Tr^{23} is the functor $\text{Tr}_{\mathcal{C}^{op}}$ applied in the 2nd and 3rd factors of the tensor product. But the latter composition can be identified with the composition of two adjunction maps $G \rightarrow G \circ F \circ G \rightarrow G$, which is equal to the identity.

(ii) The relation (2.8) shows that the left kernel of $\langle \cdot, \cdot \rangle_{G', F'}$ is trivial. On the other hand, applying (i) to the pair (G', F') we obtain

$$(\langle \cdot, \cdot \rangle_{F,G} \otimes \text{id})(y \otimes \tau_{G', F'}) = y$$

for any $y \in \text{Tr}(F) \simeq \text{Tr}(F')$. Hence, the left kernel of $\langle \cdot, \cdot \rangle_{F,G}$ is trivial. It remains to apply Lemma 2.5. \square

Remarks. 1. The above proof also shows that the tensor $\tau_{G', F'}$ is obtained from $\tau_{F,G}$ by the permutation of factors in the tensor product. This can be also seen directly from the definition.

2. In the case $F = G = \text{Id}_{\mathcal{C}}$ both $\text{Tr}(F)$ and $\text{Tr}(G)$ are identified with the Hochschild homology $HH_*(\mathcal{C})$. Then the above canonical pairing gets identified with the pairing defined in [27] (see also [22, Sec. 1.2]) using the canonical isomorphism

$$HH_*(\mathcal{C}^{op}) \simeq HH_*(\mathcal{C}) \quad (2.11)$$

(see [22, Sec. 1.1], [27, Prop. 4.6]). Note that in the case when \mathcal{C} is the dg-endomorphism algebra of a generator E of the derived category of coherent sheaves on a smooth projective variety X , and one uses E^{\vee} to construct a Morita equivalence of \mathcal{C}^{op} with the same category, the resulting isomorphisms $i : HH_*(X) \simeq HH_*(\mathcal{C})$ and $j : HH_*(X) \simeq HH_*(\mathcal{C}^{op})$ are not compatible with (2.11). As a result, the form $\langle i(a), j(b) \rangle$ on $HH_*(X)$ denoted by $\langle \cdot, \cdot \rangle_{shk}$ in [26] has somewhat different properties than $\langle \cdot, \cdot \rangle$. For example, as the proof of [26, Thm. 1] shows, this pairing is (super-)symmetric, whereas $\langle \cdot, \cdot \rangle$ is usually not. On the other hand, it seems very plausible that one can use Proposition 2.7 below to identify $\langle \cdot, \cdot \rangle$ with the Mukai pairing on $HH_*(X)$ (see [6]).

2.4. Functoriality via inverse Serre functor. The *inverse dualizing complex* of \mathcal{C} is the object

$$\Delta_{\mathcal{C}^{op}}^\vee \in \text{Per}(\mathcal{C}^{op} \otimes \mathcal{C})$$

It is well known that this kernel induces the inverse of the Serre functor on $\text{Per}(\mathcal{C})$ (cf. [12] or [28]).

Recall that for any kernel $K \in \text{Per}(\mathcal{C}^{op} \otimes \mathcal{C})$ we have a natural isomorphism

$$\text{Hom}(\Delta_{\mathcal{C}^{op}}^\vee, K) \simeq \text{Tr}_{\mathcal{C}}(K) \quad (2.12)$$

(see (1.15)). We want to rewrite the definition of the functoriality map (2.1) in terms of this isomorphism.

Thus, we assume that we have a dg-functor

$$(\Phi, \phi) : (\text{Per}_{dg}(\mathcal{C}_1), F_1) \rightarrow (\text{Per}_{dg}(\mathcal{C}_2), F_2),$$

where $\phi : \Phi \circ F_1 \rightarrow F_2 \circ \Phi$ (see Section 2.1). First, recall that there is a canonical morphism

$$\Delta_{\mathcal{C}_2^{op}}^\vee \rightarrow (\Psi' \square \Phi)(\Delta_{\mathcal{C}_1^{op}}^\vee) \simeq \Phi \circ \Delta_{\mathcal{C}_1^{op}}^\vee \circ \Psi \quad (2.13)$$

(see (1.14)). Note also that the map ϕ gives rise to a map

$$\tilde{\phi} : (\Psi' \square \Phi)(F_1) = \Phi \circ F_1 \circ \Psi \rightarrow F_2 \quad (2.14)$$

given as the composition

$$\Phi \circ F_1 \circ \Psi \xrightarrow{\phi \circ \Psi} F_2 \circ \Phi \circ \Psi \rightarrow F_2.$$

Now using isomorphism (2.12) we can give an alternative map $\text{Tr}_{\mathcal{C}_1}(F_1) \rightarrow \text{Tr}_{\mathcal{C}_2}(F_2)$ as the composition

$$\text{Hom}(\Delta_{\mathcal{C}_1^{op}}^\vee, F_1) \rightarrow \text{Hom}((\Psi' \square \Phi)(\Delta_{\mathcal{C}_1^{op}}^\vee), (\Psi' \square \Phi)(F_1)) \rightarrow \text{Hom}(\Delta_{\mathcal{C}_2^{op}}^\vee, F_2), \quad (2.15)$$

where the last arrow is induced by (2.13) and by $\tilde{\phi}$.

Proposition 2.7. *Under the isomorphism (2.12) the map (2.15) gets identified with the map (2.1).*

Proof. The map

$$\text{Hom}(\Delta_{\mathcal{C}_2^{op}}^\vee, (\Psi' \square \Phi)(F_1)) \rightarrow \text{Hom}(\Delta_{\mathcal{C}_2^{op}}^\vee, F_2)$$

induced by $\tilde{\phi}$ gets identified under the isomorphism (2.12) with the composition

$$\text{Tr}_{\mathcal{C}_2}(\Phi \circ F_1 \circ \Psi) \rightarrow \text{Tr}_{\mathcal{C}_2}(F_2 \circ \Phi \circ \Psi) \rightarrow \text{Tr}_{\mathcal{C}_2}(F_2)$$

appearing as part of the definition of (2.1). Hence, it is enough to prove the commutativity of the diagram

$$\begin{array}{ccccc} \text{Hom}(\Delta_{\mathcal{C}_1^{op}}^\vee, F_1) & \longrightarrow & \text{Hom}((\Psi' \square \Phi)(\Delta_{\mathcal{C}_1^{op}}^\vee), (\Psi' \square \Phi)(F_1)) & \longrightarrow & \text{Hom}(\Delta_{\mathcal{C}_2^{op}}^\vee, (\Psi' \square \Phi)(F_1)) \\ \downarrow & & & & \downarrow \\ \text{Tr}_{\mathcal{C}_1}(F_1) & \longrightarrow & \text{Tr}_{\mathcal{C}_1}(F_1 \circ \Psi \circ \Phi) & \longrightarrow & \text{Tr}_{\mathcal{C}_2}(\Phi \circ F_1 \circ \Psi) \end{array} \quad (2.16)$$

Using the adjointness of the pair $(\Psi' \square \Phi, \Phi' \square \Psi)$ the first arrow in the top row can be identified with the map

$$\text{Hom}(\Delta_{\mathcal{C}_1^{op}}^\vee, F_1) \rightarrow \text{Hom}(\Delta_{\mathcal{C}_1^{op}}^\vee, (\Phi' \Psi' \square \Psi \Phi)(F_1)).$$

Next, by Lemma 1.3(ii) the second arrow in the top row is equal to the following composition (where we denoted $X = (\Psi' \square \Phi)(F_1)$):

$$\text{Hom}((\Psi' \square \Phi)(\Delta_{\mathcal{C}_1^{op}}^\vee), X) \rightarrow \text{Hom}(\Delta_{\mathcal{C}_1^{op}}^\vee, (\Phi' \square \Psi)(X)) \xrightarrow{t(\Phi, \Psi)(X)} \text{Hom}(\Delta_{\mathcal{C}_1^{op}}^\vee, X).$$

Therefore, by Lemma 1.3(i), the composition of arrows in the top row of (2.16) can be identified with the composition

$$\text{Tr}(F_1) \rightarrow \text{Tr}((\Phi' \Psi' \square \Psi \Phi)(F_1)) \xrightarrow{\sim} \text{Tr}(\Psi \Phi F_1 \Psi \Phi) \xrightarrow{\sim} \text{Tr}(\Phi \Psi \Phi F_1 \Psi) \rightarrow \text{Tr}(\Phi F_1 \Psi).$$

Note that the composition of the first two arrows is equal to the composition of the following two maps induced by adjunction:

$$\mathrm{Tr}(F_1) \rightarrow \mathrm{Tr}(F_1 \circ \Psi \circ \Phi) \rightarrow \mathrm{Tr}(\Psi \Phi F_1 \Psi \Phi).$$

Hence, the commutativity of (2.16) follows from the commutativity of the outer trapezoid in the following diagram

$$\begin{array}{ccccc} & & \mathrm{Tr}(\Psi \Phi F_1 \Psi \Phi) & \longrightarrow & \mathrm{Tr}(\Phi \Psi \Phi F_1 \Psi) \\ & \uparrow & & & \uparrow \\ & \mathrm{Tr}(F_1 \Psi \Phi) & \longrightarrow & \mathrm{Tr}(\Phi F_1 \Psi) & \xrightarrow{=} \mathrm{Tr}(\Phi F_1 \Psi) \end{array}$$

in which the vertical arrows are induced by adjunction and the horizontal arrows in the left square are isomorphisms of the form $\mathrm{Tr}(X \circ \Phi) \xrightarrow{\sim} \mathrm{Tr}(\Phi \circ X)$. The commutativity of the triangle in this diagram follows from the fact that the composition $\Phi \rightarrow \Phi \Psi \Phi \rightarrow \Phi$ of the maps induced by adjunction is equal to the identity. \square

In the situation when F_1 and F_2 are the identity functors we derive the following

Corollary 2.8. *Let $\Phi : \mathrm{Per}_{dg}(\mathcal{C}_1) \rightarrow \mathrm{Per}_{dg}(\mathcal{C}_2)$ be a dg-functor. Then the induced map*

$$\Phi_* : HH_*(\mathcal{C}_1) \rightarrow HH_*(\mathcal{C}_2)$$

defined as in Section 2.1, is equal to the composition

$$\mathrm{Hom}(\Delta_{\mathcal{C}_1^{op}}^\vee, \Delta_{\mathcal{C}_1}) \rightarrow \mathrm{Hom}(\Phi \circ \Delta_{\mathcal{C}_1^{op}}^\vee \circ \Psi, \Phi \circ \Psi) \rightarrow \mathrm{Hom}(\Delta_{\mathcal{C}_2^{op}}^\vee, \Delta_{\mathcal{C}_2}),$$

where the second arrow is induced by (2.13) and by the adjunction map $\Phi \circ \Psi \rightarrow \Delta_{\mathcal{C}_2}$.

Remark. One can deduce from the above corollary that our definition of functoriality maps on Hochschild homology is equivalent to the one given in [6] and [2, Appendix].

Recall that *Hochschild cohomology* of \mathcal{C} is defined by

$$HH^*(\mathcal{C}) = \mathrm{Hom}_{\mathcal{C} \otimes \mathcal{C}^{op}}(\Delta_{\mathcal{C}}, \Delta_{\mathcal{C}}).$$

Note that for any autoequivalence $F : \mathrm{Per}(\mathcal{C}) \rightarrow \mathrm{Per}(\mathcal{C})$ (given by a kernel) we have a natural isomorphism

$$HH^*(\mathcal{C}) \xrightarrow{\sim} \mathrm{Hom}(F, F)$$

sending α to $F \circ \alpha$. Applying this to $F = \Delta_{\mathcal{C}^{op}}^\vee$ we obtain an isomorphism

$$HH^*(\mathcal{C}) \simeq \mathrm{Hom}(\Delta_{\mathcal{C}^{op}}^\vee, \Delta_{\mathcal{C}^{op}}^\vee) \simeq \mathrm{Tr}_{\mathcal{C}}(\Delta_{\mathcal{C}^{op}}^\vee). \quad (2.17)$$

Now if $\Phi : \mathrm{Per}(\mathcal{C}_1) \rightarrow \mathrm{Per}(\mathcal{C}_2)$ is an equivalence (given by a kernel) then there is an induced isomorphism

$$\Phi_* : HH^*(\mathcal{C}_1) \rightarrow HH^*(\mathcal{C}_2). \quad (2.18)$$

We are going to show that this map appears as one of our functoriality maps. Since Φ is an equivalence, we have a natural isomorphism

$$\tilde{\phi} : \Phi \circ \Delta_{\mathcal{C}_1^{op}}^\vee \circ \Phi^{-1} \xrightarrow{\sim} \Delta_{\mathcal{C}_2^{op}}^\vee,$$

and hence the induced isomorphism

$$\phi : \Phi \circ \Delta_{\mathcal{C}_1^{op}}^\vee \xrightarrow{\sim} \Delta_{\mathcal{C}_2^{op}}^\vee \circ \Phi. \quad (2.19)$$

Corollary 2.9. *The map (2.18) gets identified under the isomorphisms (2.17) with*

$$(\Phi, \phi)_* : \mathrm{Tr}_{\mathcal{C}_1}(\Delta_{\mathcal{C}_1^{op}}^\vee) \rightarrow \mathrm{Tr}_{\mathcal{C}_2}(\Delta_{\mathcal{C}_2^{op}}^\vee).$$

Proof. By Proposition 2.7, it is enough to prove the commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{Hom}(\Delta_{C_1}, \Delta_{C_1}) & \xrightarrow{\alpha \mapsto \Phi \circ \alpha \circ \Phi^{-1}} & \mathrm{Hom}(\Delta_{C_2}, \Delta_{C_2}) \\
\downarrow \sim & & \downarrow \sim \\
\mathrm{Hom}(\Delta_{C_1}^{\vee op}, \Delta_{C_1}^{\vee op}) \rightarrow \mathrm{Hom}(\Phi \circ \Delta_{C_1}^{\vee op} \circ \Phi^{-1}, \Phi \circ \Delta_{C_1}^{\vee op} \circ \Phi^{-1}) & \rightarrow & \mathrm{Hom}(\Delta_{C_2}^{\vee op}, \Delta_{C_2}^{\vee op})
\end{array}$$

where the vertical arrows are isomorphisms (2.17) and the second arrow in the bottom row is induced by the isomorphism $\tilde{\phi}$. Upon unraveling the definitions this becomes a tautology. \square

3. LEFSCHETZ TYPE FORMULAS

3.1. Categorical version of holomorphic Lefschetz formula. Now let A and B be a pair of objects of $\mathrm{Per}(\mathcal{C})$ equipped with morphisms $\alpha : A \rightarrow F(A)$ and $\beta : F(B) \rightarrow B$. Let G be the right quasi-adjoint to F , and let F' be the functor associated with the kernel $K' := K \circ \sigma$ (see Section 1.2).

Theorem 3.1. *In this situation the formula (0.2) holds, where τ_F^A (resp., τ_G^B) is the generalized boundary bulk map of Section 2.2, and $\langle \cdot, \cdot \rangle_{F,G}$ is the canonical pairing of Section 2.3.*

Proof. We would like to apply Lemma 2.4 to the dg-functor (2.7), extending $\mathrm{Tr}_{\mathcal{C}}^{dg}$, from $(\mathrm{Per}_{dg}(\mathcal{C}^{op} \otimes \mathcal{C}), F' \square G)$ to $\mathrm{Per}_{dg}(k, \mathrm{Id})$.

By Lemma 1.2, the map $\alpha : A \rightarrow F(A)$ in $\mathrm{Per}(\mathcal{C})$ induces a morphism $\alpha' : A^{\vee} \rightarrow F'(A^{\vee})$ in $\mathrm{Per}(\mathcal{C}^{op})$. Also, $\beta : F(B) \rightarrow B$ induces a map $\tilde{\beta} : B \rightarrow G(B)$. Hence, we obtain a morphism

$$\alpha' \otimes \tilde{\beta} : A^{\vee} \boxtimes B \rightarrow (F' \square G)(A^{\vee} \boxtimes B).$$

Note that

$$\tau_{F' \square G}^{A^{\vee} \boxtimes B} = \tau_{F'}^{A^{\vee}} \otimes \tau_G^B.$$

Also, for a graded vector space V the map

$$\tau_{\mathrm{Id}}^V : \mathrm{Hom}(V, V) \rightarrow k$$

is the usual supertrace. Hence, Lemma 2.4 gives an equality

$$\mathrm{str}(f) = \langle \tau_{F'}^{A^{\vee}}(\alpha'), \tau_G^B(\tilde{\beta}) \rangle_{F,G}, \quad (3.1)$$

where $f = (\mathrm{Tr}_{\mathcal{C}}^{dg}, t(F, G))_*^{A^{\vee} \boxtimes B}(\alpha' \otimes \tilde{\beta})$ is the composition

$$\mathrm{Hom}(A, B) \simeq \mathrm{Tr}_{\mathcal{C}}(A^{\vee} \boxtimes B) \xrightarrow{\mathrm{Tr}_{\mathcal{C}}(\alpha' \otimes \tilde{\beta})} \mathrm{Tr}_{\mathcal{C}}(F'(A^{\vee}) \boxtimes G(B)) \xrightarrow{t(F,G)} \mathrm{Tr}_{\mathcal{C}}(A^{\vee} \boxtimes B) \simeq \mathrm{Hom}(A, B).$$

Since $\tau_{F'}^{A^{\vee}}(\alpha') = \tau_F^A(\alpha)$ by Lemma 2.3, the right-hand side of (3.1) coincides with the right-hand side of (0.2). It remains to check that f is equal to the endomorphism $(F, \alpha, \beta)_*$ of $\mathrm{Hom}(A, B)$ (see (0.1)). We can rewrite the definition of $(F, \alpha, \beta)_*$ as the following composition:

$$\mathrm{Hom}(A, B) \simeq \mathrm{Tr}_{\mathcal{C}}(A^{\vee} \boxtimes B) \xrightarrow{s} \mathrm{Tr}_{\mathcal{C}}(F(A)^{\vee} \boxtimes F(B)) \xrightarrow{\mathrm{Tr}_{\mathcal{C}}(\alpha^{\vee} \otimes \beta)} \mathrm{Tr}_{\mathcal{C}}(A^{\vee} \boxtimes B) \simeq \mathrm{Hom}(A, B),$$

where s is induced by the isomorphism $F(A)^{\vee} \simeq G'(A^{\vee})$ (see (1.5)) and by a natural morphism of functors

$$s(G, F) : \mathrm{Tr}_{\mathcal{C}} \rightarrow \mathrm{Tr}_{\mathcal{C}} \circ (G' \square F).$$

This morphism of functors is constructed similarly to $t(F, G)$ using the map of kernels (1.4). Now the required equality $f = (F, \alpha, \beta)_*$ follows from the commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{Tr}_{\mathcal{C}}(A^\vee \boxtimes B) & \xrightarrow{\mathrm{Tr}(\alpha' \otimes \tilde{\beta})} & \mathrm{Tr}_{\mathcal{C}}(F'(A^\vee) \boxtimes G(B)) \\
\downarrow s(G, F) & & \downarrow s(G, F) \\
\mathrm{Tr}_{\mathcal{C}}(G'(A^\vee) \boxtimes F(B)) & \xrightarrow{\mathrm{Tr}(G'(\alpha') \otimes F(\tilde{\beta}))} & \mathrm{Tr}_{\mathcal{C}}(G'F'(A^\vee) \boxtimes FG(B)) \\
\downarrow & & \downarrow \\
\mathrm{Tr}_{\mathcal{C}}(F(A)^\vee \boxtimes F(B)) & \xrightarrow{\mathrm{Tr}_{\mathcal{C}}(\alpha^\vee \otimes \beta)} & \mathrm{Tr}_{\mathcal{C}}(A^\vee \boxtimes B)
\end{array}$$

where the vertical arrows in the lower square are induced by the isomorphism $G'(A^\vee) \simeq F(A)^\vee$ and by the adjunction morphisms $G'F' \rightarrow \mathrm{Id}$ and $FG \rightarrow \mathrm{Id}$. Note that the commutativity of the lower square follows from Lemma 1.2(ii). \square

Example. The classical setup is recovered if we take \mathcal{C} to be a dg-enhancement of the derived category $D^b(M)$ of coherent sheaves on a smooth projective variety M and $F = f^*$, where $f : M \rightarrow M$ is an endomorphism. The category of kernels in this case can be identified with $D^b(M \times M)$, so that the functor $\mathrm{Tr} : D^b(M \times M) \rightarrow \mathrm{Per}(k)$ sends K to $H^*(M, \Delta^*(K))$. In particular, the space

$$T_f := \mathrm{Tr}(f^*) = \mathrm{Tr}(f_*) = H^*(M, \Delta^*(\mathcal{O}_{\Gamma_f}))$$

corresponds to considering the derived intersection of the graph Γ_f with the diagonal. In the case when the intersection is transversal we have

$$T_f = \bigoplus_{f(x)=x} k.$$

If V is a vector bundle (or a complex of vector bundles) on M then applying $(f \times \mathrm{id}_M)^*$ to the natural map $(V^\vee \boxtimes V) \rightarrow \mathcal{O}_\Delta$ and then restricting to the diagonal we get a map

$$\tau_f^V : \mathrm{Hom}(V, f_* V) = \mathrm{Hom}(f^* V, V) \rightarrow T_f$$

which is exactly the map used in (0.2). For example, for $V = \mathcal{O}_X$ we get $\tau_f^{\mathcal{O}_X}(\mathrm{id}) = \mathbf{1} \in T_f$, the element obtained from the projection $\mathcal{O}_X \rightarrow \mathcal{O}_{\Gamma_f}$. In the case of transversal fixed points we have

$$\tau_f^V(\alpha) = \sum_{f(x)=x} \mathrm{tr}(\alpha, V|_x) \cdot \delta_x,$$

where (δ_x) is the natural basis of T_f indexed by fixed points. Finally, we claim that in the case of transversal intersection the canonical form $\langle \cdot, \cdot \rangle$ on T_f is given by

$$\langle \delta_x, \delta_x \rangle = \frac{1}{\det(\mathrm{id} - d_x f)} \quad \text{and} \quad \langle \delta_x, \delta_y \rangle = 0 \quad \text{for } x \neq y,$$

where $d_x f : T_x M \rightarrow T_x M$ is the tangent map to f at x . Indeed, the fact that the basis (δ_x) is orthogonal can be deduced from Lemma 2.6. To get the formula for $\langle \delta_x, \delta_x \rangle$, where $f(x) = x$, we observe that $\tau_f^{\mathcal{O}_x} = c_x \cdot \delta_x$ for some $c_x \in k$, and apply the formula (0.2) to the pairs $(\mathcal{O}_M, \mathcal{O}_x)$ and to $(\mathcal{O}_x, \mathcal{O}_x)$. In the former case we get

$$1 = \langle \mathbf{1}, c_x \delta_x \rangle = c_x \cdot \langle \delta_x, \delta_x \rangle,$$

and in the latter case we get

$$\mathrm{str}(F_*, \mathrm{Ext}^*(\mathcal{O}_x, \mathcal{O}_x)) = c_x^2 \cdot \langle \delta_x, \delta_x \rangle.$$

Hence,

$$c_x = \langle \delta_x, \delta_x \rangle^{-1} = \text{str}(f_*, \text{Ext}^*(\mathcal{O}_x, \mathcal{O}_x)).$$

But $\text{Ext}^*(\mathcal{O}_x, \mathcal{O}_x)$ is the exterior algebra of $\text{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) \simeq T_x M$ and the action of F on $\text{Ext}^1(\mathcal{O}_x, \mathcal{O}_x)$ is precisely the tangent map $d_x f$. Since F_* is compatible with the algebra structure, we obtain

$$\text{str}(F_*, \text{Ext}^*(\mathcal{O}_x, \mathcal{O}_x)) = \det(\text{id} - d_x f).$$

Thus, our formula in this case becomes the classical holomorphic Lefschetz formula

$$\text{str}(f, H^*(M, V)) = \sum_{f(x)=x} \text{tr}(f, V|_p) \cdot \det(\text{id} - d_x f)^{-1}.$$

Using the Lefschetz formula (0.2) we can give some criteria for non-vanishing of the space $\text{Tr}_{\mathcal{C}}(F)$ (which is roughly analogous to the existence of a fixed point of an endomorphism).

Recall that an object A of a k -linear triangulated category is called *exceptional* if $\text{Hom}^i(A, A) = 0$ for $i \neq 0$ and $\text{Hom}^0(A, A) = k$. An object A is called *n-spherical* if $\text{Hom}^i(A, A) = 0$ for $i \neq 0, n$ and $\text{Hom}^0(A, A) = \text{Hom}^n(A, A)$.

Corollary 3.2. *One has $\text{Tr}_{\mathcal{C}}(F) \neq 0$ if one of the following conditions holds:*

- (i) *there exists an exceptional object $A \in \text{Per}(\mathcal{C})$ such that $F(A) \simeq A$;*
- (ii) *there exists an n-spherical object $A \in \text{Per}(\mathcal{C})$, where n is even, such that $F(A) \simeq A$, and in addition $F^N \simeq \text{Id}$, where N is odd;*
- (iii) *there exists an object $A \in \text{Per}(\mathcal{C})$ with $\text{Hom}(A, A) \simeq \bigwedge_k^*(\text{Hom}^1(A, A))$, such that $F(A) \simeq A$ and the endomorphism F_* of $\text{Hom}^1(A, A)$ satisfies $\det(\text{id} - F_*) \neq 0$.*

Proof. We apply (0.2) to $A = B$, with $\alpha : A \rightarrow F(A)$ some isomorphism and $\beta = \alpha^{-1}$. We have to show that in all three cases we have $\text{str}(F_*, \text{Hom}(A, A)) \neq 0$.

In case (i) we have $\text{Hom}(A, A) = k$ and the induced map F_* on it is identity, so $\text{str}(F_*, \text{Hom}(A, A)) = 1$.

In case (ii) F_* acts as identity on $\text{Hom}^0(A, A) = k \cdot \text{id}_A$ and as some N th root of unity ζ on $\text{Hom}^n(A, A) \simeq k$. Since n is even, we obtain

$$\text{str}(F_*, \text{Hom}(A, A)) = 1 + \zeta$$

which is nonzero since N is odd.

Finally, in case (iii) we use the fact that F_* is an endomorphism of the algebra $\text{Hom}(A, A) \simeq \bigwedge_k^*(\text{Hom}^1(A, A))$, so we get

$$\text{str}(F_*, \text{Hom}(A, A)) = \det(\text{id} - F_*) \neq 0.$$

□

Note that the property (iii) in the above Corollary is modeled on the properties of the pair (f^*, \mathcal{O}_x) for a transversal fixed point x of an endomorphism f .

3.2. Lefschetz reciprocity. As before, we assume that \mathcal{C} is a dg-category such that $\text{Per}_{dg}(\mathcal{C})$ is saturated, and F is a dg-endofunctor of $\text{Per}_{dg}(\mathcal{C})$. Now let Ψ be another dg-endofunctor of $\text{Per}_{dg}(\mathcal{C})$ together with a morphism

$$f : F \circ \Psi \rightarrow \Psi \circ F$$

Then we have the induced endomorphism $(F, f)_*$ of $\text{Tr}(\Psi)$. Let G be right adjoint to F . Then we have the induced map

$$\psi : \Psi \circ G \rightarrow G \circ F \circ \Psi \circ G \xrightarrow{G \circ f \circ G} G \circ \Psi \circ F \circ G \rightarrow G \circ \Psi,$$

and hence the induced endomorphism $(\Psi, \psi)_*$ of $\text{Tr}(G)$. We have the following *Lefschetz reciprocity* formula.

Theorem 3.3. *In the above situation*

$$\text{str}((F, f)_*, \text{Tr}(\Psi)) = \text{str}((\Psi, \psi)_*, \text{Tr}(G)). \quad (3.2)$$

Proof. The idea is to apply formula (0.2) to the category $\mathcal{C}^{op} \otimes \mathcal{C}$, equipped with the endofunctor $G' \square F$, taking as B the kernel representing Ψ and

$$A = \Delta_{\mathcal{C}^{op}}^\vee,$$

the inverse dualizing complex (see section 2.4).

Note that we have

$$\text{Tr}_{\mathcal{C}^{op} \otimes \mathcal{C}}(G' \square F) \simeq \text{Tr}_{\mathcal{C}^{op}}(G') \otimes \text{Tr}_{\mathcal{C}}(F) \simeq \text{Tr}(G) \otimes \text{Tr}(F).$$

Also, the right quasi-adjoint to $G' \square F$ is $F' \square G$ (by Lemma 1.1).

We will apply (0.2) to certain natural morphisms

$$\alpha : \Delta_{\mathcal{C}^{op}}^\vee \rightarrow (G' \square F)(\Delta_{\mathcal{C}^{op}}^\vee) \text{ and}$$

$$\beta : (G' \square F)(B) \rightarrow B$$

that we are going to define presently. Namely, using the isomorphism of Lemma 1.2(i), α corresponds to a morphism

$$\alpha' : \Delta_{\mathcal{C}^{op}} \rightarrow (G' \square F)'(\Delta_{\mathcal{C}^{op}}) = (G \square F')(\Delta_{\mathcal{C}^{op}}),$$

and we take

$$\alpha' := \gamma_{G', F'}$$

(see (1.13)). On the other hand, β corresponds to the morphism of functors

$$F \circ \Psi \circ G \xrightarrow{f \circ G} \Psi \circ F \circ G \rightarrow \Psi.$$

It follows that the map

$$\tilde{\beta} : B \rightarrow (F' \square G)(B),$$

obtained from β by adjunction of the functors $(G' \square F, F' \square G)$, corresponds to the morphism of functors

$$\Psi \rightarrow \Psi \circ G \circ F \xrightarrow{\psi \circ F} G \circ \Psi \circ F.$$

It follows from Proposition (2.7) that the endomorphism $(G' \square F, \alpha, \beta)_*$ of $\text{Hom}(A, B) = \text{Hom}(\Delta_{\mathcal{C}^{op}}^\vee, B) = \text{Tr}(\Psi)$ can be identified with the endomorphism $(F, f)_*$ of $\text{Tr}(\Psi)$.

Thus, the formula (0.2) combined with Lemma 2.3 gives

$$\text{str}((F, f)_*, \text{Tr}(\Psi)) = \langle \tau_{G \square F'}^{\Delta_{\mathcal{C}^{op}}}(\gamma_{G', F'}), \tau_{F' \square G}^B(\tilde{\beta}) \rangle = \langle \tau_{G', F'}, \tau_{F' \square G}^B(\tilde{\beta}) \rangle.$$

We claim that

$$\tau_{F' \square G}^B(\tilde{\beta}) = (\text{id} \otimes (\Psi, \psi)_*)(\tau_{F, G}), \quad (3.3)$$

where

$$\psi : \Psi \circ G \rightarrow G \circ \Psi$$

is induced by f . Indeed, this follows from Lemma 2.4 since $B = (\text{Id} \square \Psi)(\Delta_{\mathcal{C}})$.

Applying Lemma 3.4 below we get

$$\langle \tau_{G', F'}(\text{id} \otimes (\Psi, \psi)_*)(\tau_{F, G}) \rangle = \text{str}((\Psi, \psi)_*, \text{Tr}(G))$$

It remains to observe that starting with the data (F, Ψ, f) we can recover the data (Φ, F, ϕ) . Hence, we can consider (F, Ψ, f) as the primary data. Now (3.2) is obtained from the above equality by a change of notation. \square

Lemma 3.4. *Let V and W be finite-dimensional supervector spaces equipped with a perfect (even) pairing*

$$\langle \cdot, \cdot \rangle_{V,W} : V \otimes W \rightarrow k.$$

Let $\tau \in V \otimes W$ be the corresponding Casimir element, and let $\tau' = \sigma(\tau) \in W \otimes V$. Consider the induced pairing

$$\langle \cdot, \cdot \rangle_{V \otimes W, W \otimes V} : (V \otimes W) \otimes (W \otimes V) \rightarrow k$$

given by

$$\langle v \otimes w, w' \otimes v' \rangle_{V \otimes W, W \otimes V} = \pm \langle v, w' \rangle_{V,W} \cdot \langle v', w \rangle_{V,W}$$

with the sign given by the Koszul rule. Then for any even automorphism A of V one has

$$\langle \tau, (\text{id} \otimes A)(\tau') \rangle_{V \otimes W, W \otimes V} = \text{str}(A).$$

In particular,

$$\langle \tau, \tau' \rangle_{V \otimes W, W \otimes V} = \text{sdim}(V).$$

Proof. Let (v_i) be a homogeneous basis in V and let (w_i) be the dual basis of W , so that $\langle v_i, w_j \rangle = \delta_{ij}$. Then

$$\tau = \sum_i (-1)^{\deg(v_i)} v_i \otimes w_i \quad \text{and} \quad \tau' = \sum_i w_i \otimes v_i.$$

It is enough to prove the formula for A such that $A(v_i) = v_j$ and $A(v_l) = 0$ for $l \neq i$. Then we have

$$\langle \tau, (\text{id} \otimes A)(\tau') \rangle_{V \otimes W, W \otimes V} = \langle (-1)^{\deg(v_i)} v_i \otimes w_i, w_i \otimes v_j \rangle_{V \otimes W, W \otimes V} = \delta_{ij} (-1)^{\deg(v_i)} = \text{str}(A).$$

□

Remark. In the particular case $(\Psi, \psi) = (\text{Id}_C, \text{id})$ the formula (3.2) gives (0.3). Note however, that the argument of [17] for the proof of (0.3) is more direct.

Corollary 3.5. *Keep the assumptions of Theorem 3.3. Set $\tilde{\Psi} = \Psi \circ F$ and let*

$$\tilde{f} : F \circ \tilde{\Psi} = F \circ \Psi \circ F \rightarrow \Psi \circ F \circ F = \tilde{\Psi} \circ F$$

be the map $f \circ F$. Then

$$\text{str}((F, f)_*, \text{Tr}(\Psi)) = \text{str}((F, \tilde{f}), \text{Tr}(\tilde{\Psi})). \quad (3.4)$$

A similar results holds for $F \circ \Psi$ instead of $\Psi \circ F$.

Proof. By Theorem 3.3, the equality (3.4) is equivalent to

$$\text{str}((\Psi, \psi)_*, \text{Tr}(G)) = \text{str}((\tilde{\Psi}, \tilde{\psi})_*, \text{Tr}(G)),$$

where

$$\tilde{\psi} : \tilde{\Psi} \circ G \rightarrow G \circ \tilde{\Psi}$$

is defined similarly to ψ , starting from \tilde{f} . We claim that in fact

$$(\Psi, \psi)_* = (\tilde{\Psi}, \tilde{\psi})_*. \quad (3.5)$$

Let us first calculate the morphism $\tilde{\psi}$. We have a commutative diagram

$$\begin{array}{ccccc} \Psi \circ F \circ G & \longrightarrow & G \circ F \circ \Psi \circ F \circ G & \xrightarrow{G \circ f \circ F \circ G} & G \circ \Psi \circ F \circ F \circ G \\ \downarrow & & \downarrow & & \downarrow \\ \Psi & \longrightarrow & G \circ F \circ \Psi & \xrightarrow{G \circ f} & G \circ \Psi \circ F \end{array}$$

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Here and in the diagrams below unnamed arrows are induced by adjunction morphisms. By definition, $\tilde{\psi}$ is given by the composition of the arrows in the top row followed by the right vertical arrow. Hence, it can also be computed as the composition of the arrows in the bottom row with the left vertical arrow. Now the commutative diagram

$$\begin{array}{ccccc}
\Psi & \xrightarrow{\quad} & G \circ F \circ \Psi & \xrightarrow{G \circ f} & G \circ \Psi \circ F \\
\downarrow & & \downarrow & & \downarrow \\
\Psi \circ G \circ F & \xrightarrow{\quad} & G \circ F \circ \Psi \circ G \circ F & \xrightarrow{G \circ f \circ G \circ F} & G \circ \Psi \circ F \circ G \circ F \xrightarrow{\quad} G \circ \Psi \circ F
\end{array}$$

$\searrow \text{id}$

together with the definition of ψ , shows that $\tilde{\psi}$ is equal to the composition

$$\Psi \circ F \circ G \xrightarrow{\Psi \circ c} \Psi \circ G \circ F \xrightarrow{\psi \circ F} G \circ \Psi \circ F,$$

where

$$c : F \circ G \rightarrow \text{Id} \rightarrow G \circ F$$

is the composition of two adjunction morphisms. Hence, by Lemma 2.1, (3.5) would follow once we prove that the map

$$(F, c)_* : \text{Tr}(G) \rightarrow \text{Tr}(G)$$

is the identity map. Recall (see Sec. 2.1) that the map $(F, c)_*$ is given by the composition

$$\text{Tr}(G) \rightarrow \text{Tr}(G \circ F \circ G) \xrightarrow{\sigma} \text{Tr}(F \circ G \circ G) \xrightarrow{\text{Tr}(c \circ G)} \text{Tr}(G \circ F \circ G) \rightarrow \text{Tr}(G),$$

Here and below arrows marked by σ are induced by the isomorphism (1.10). By definition of the morphism c this is equal to the composition

$$\text{Tr}(G) \rightarrow \text{Tr}(G \circ F \circ G) \xrightarrow{\sigma} \text{Tr}(F \circ G \circ G) \rightarrow \text{Tr}(G) \rightarrow \text{Tr}(G \circ F \circ G) \rightarrow \text{Tr}(G). \quad (3.6)$$

Note that the composition of the last two arrows is the identity map, which together with the commutative triangle

$$\begin{array}{ccc}
\text{Tr}(G \circ F \circ G) & & \\
\downarrow \sigma & \searrow & \\
\text{Tr}(F \circ G \circ G) & \longrightarrow & \text{Tr}(G)
\end{array}$$

shows that the composition (3.6) is equal to the identity, i.e., $(F, c)_* = \text{id}$. \square

Consider again the situation when \mathcal{C} is a dg-enhancement of $D^b(M)$, where M is a smooth projective variety over k , $F = f^*$ and $\Psi(\mathcal{F}) = V \otimes_{\mathcal{O}_M} \mathcal{F}$, where V is a bounded complex of vector bundles on M . Thus, the kernel on $M \times M$ corresponding to Ψ is $\Delta_* V$, where $\Delta : V \rightarrow V \times V$ is the diagonal embedding. A map $F \circ \Psi \rightarrow \Psi \circ F$ is the same as a map $\alpha : f^* V \rightarrow V$, and there is a natural isomorphism

$$\text{Tr}(\Psi) \simeq H^*(M, \Delta^* \Delta_* V) \simeq H^*(M, \Omega_M^* \otimes V).$$

The map $(F, \alpha)_* : \text{Tr}(\Psi) \rightarrow \text{Tr}(\Psi)$ induced by α is the composition

$$\alpha_* : H^*(M, \Omega_M^* \otimes V) \xrightarrow{f^*} H^*(M, \Omega_M^* \otimes f^* V) \xrightarrow{\alpha} H^*(M, \Omega_M^* \otimes V).$$

On the other hand, the functor $F = f^*$ is given by the kernel $(f, \text{id}_M)_* \mathcal{O}_M$ on $M \times M$, so $\Psi \circ F$ is given by the kernel $(f, \text{id}_M)_* V$. Hence,

$$\text{Tr}(\Psi \circ F) \simeq H^*(M, \Delta^*(f, \text{id}_M)_* V) \simeq H^*(M, i_* i^* V) \simeq H^*(M^f, i^* V),$$

where M^f is the *derived intersection* of Γ_f with the diagonal in $M \times M$, fitting into the derived cartesian diagram

$$\begin{array}{ccc} M^f & \xrightarrow{i} & M \\ \downarrow i & (\text{id}_M, f) & \downarrow \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

Corollary 3.5 leads in this situation to the following result conjectured by Frenkel-Ngô (see [11, Conj. 6.2]).

Theorem 3.6. *Let V be a bounded complex of vector bundle on a smooth projective variety M over k , $f : M \rightarrow M$ and endomorphism. Then for any morphism $\alpha : f^*V \rightarrow V$ in $D^b(M)$ one has*

$$\text{str}(\alpha_*, H^*(M, \Omega_M^* \otimes V)) = \text{str}((i^*\alpha)_* : H^*(M^f, i^*V)), \quad (3.7)$$

where the map $(i^*\alpha)_*$ is given by

$$H^*(M^f, i^*V) \simeq H^*(M^f, i^*f^*V) \xrightarrow{i^*\alpha} H^*(M^f, i^*V)$$

Remarks. 1. In the case of transversal intersection the formula (3.7) becomes

$$\text{str}(f, H^*(M, \Omega_M^* \otimes V)) = \sum_{f(x)=x} \text{tr}(f, V|_p)$$

which follows also from the usual holomorphic Lefschetz formula (see Example 3.1).

2. In the formula of [11, Conj. 6.2] the right-hand side apparently has a typo: it should have i^*V rather than $i^*\Delta_*(V)$ in order to be correct even in the transversal case.

3. Theorem 3.6 and its proof should generalize to an appropriate class of derived algebraic stacks, as envisioned in [11].

3.3. Lefschetz formula in Hochschild cohomology. Now let (\mathcal{C}, F) be as above and assume in addition that F is an autoequivalence, so that $G = F^{-1}$. Let us take $\Psi = \mathcal{S}^{-1}$, the inverse Serre functor given by the kernel $\Delta_{\mathcal{C}^{op}}^\vee$, and use the canonical morphism $f : F \circ \mathcal{S}^{-1} \rightarrow \mathcal{S}^{-1} \circ F$ (see (2.19)). As we have seen in Corollary 2.9, the map $(F, f)_*$ gets identified with the natural endomorphism F_* of the Hochschild cohomology $HH^*(\mathcal{C}) \simeq \Delta_{\mathcal{C}^{op}}^\vee$. Thus, Theorem (3.3) gives a formula for the supertrace $\text{str}(F_*, HH^*(\mathcal{C}))$. We will rewrite this formula in a slightly different way.

Note that since F' is an autoequivalence, we have two non-degenerate pairings $\text{Tr}(F') \otimes \text{Tr}(G) \simeq \text{Tr}(F) \otimes \text{Tr}(G') \rightarrow k$: $\langle \cdot, \cdot \rangle_{F, G}$ and $\langle \cdot, \cdot \rangle_{F', G'}$ (see Section 2.3). Let $A_F : \text{Tr}(G) \rightarrow \text{Tr}(G)$ be the automorphism such that

$$\langle x, y \rangle_{F, G} = \langle A_F(x), y \rangle_{F', G'}.$$

Theorem 3.7. *For a tensor autoequivalence F of $\text{Per}(\mathcal{C})$ one has*

$$\text{str}(F_*, HH^*(\mathcal{C})) = \text{str}((\mathcal{S}^{-1}, s)_*, \text{Tr}_{\mathcal{C}}(G)),$$

where

$$s : \mathcal{S}^{-1} \circ G \rightarrow G \circ \mathcal{S}^{-1}$$

is induced by the isomorphism (2.19). Also, we have

$$(\mathcal{S}^{-1}, s)_* = A_F.$$

Proof. By Theorem (3.3), we only have to check that $(\mathcal{S}^{-1}, s)_* = A_F$. We are going to deduce this from (3.3). Note that in our case $B = \Delta_{\mathcal{C}^{op}}^\vee$, $\Psi = \mathcal{S}^{-1}$, and as the proof of Theorem 3.3 shows, $\tilde{\beta}$ is the map

$$\mathcal{S}^{-1} \rightarrow \mathcal{S}^{-1} \circ G \circ F \xrightarrow{s \circ F} G \circ \mathcal{S}^{-1} \circ F,$$

or on the level of kernels,

$$\Delta_{\mathcal{C}^{op}}^\vee \rightarrow (F' \square G)(\Delta_{\mathcal{C}^{op}}^\vee).$$

It is easy to see that this map corresponds under the isomorphism of Lemma 1.2(i) to $\gamma_{F', G'}$. Hence, by Lemma 2.3 we obtain

$$\tau_{F' \square G}^B(\tilde{\beta}) = \tau_{F' \square G'}^{\Delta_{\mathcal{C}}}(\gamma_{F', G'}) = \tau_{F', G'}.$$

Thus, (3.3) gives the relation

$$\tau_{F', G'} = (\text{id} \otimes (\mathcal{S}^{-1}, s)_*)(\tau_{F, G})$$

which is equivalent to what we needed to check. \square

In the case $F = \text{Id}$, the above theorem gives the following formula.

Corollary 3.8. *One has the equality*

$$\text{sdim } HH^*(\mathcal{C}) = \text{str}((\mathcal{S}^{-1})_*, HH_*(\mathcal{C}))$$

in k .

Example. In the situation when \mathcal{C} is the dg-version of the derived category of coherent sheaves on a smooth complex projective variety X the formula of Corollary 3.8 becomes

$$\text{sdim } HH^*(X) = (-1)^n \text{sdim } HH_*(X),$$

where $n = \dim X$. Indeed, due to the form of the Serre functor in this case, the operator $(-1)^n \mathcal{S}_*^{-1}$ on $HH_*(X)$ is upper triangular, so its supertrace is equal to $\text{sdim } HH_*(X)$. Note that the above equality follows also from the well-known identity

$$\text{sdim } HH^*(X) = c_n(\Omega_X^1) = (-1)^n c_n(T_X) = (-1)^n \text{sdim } HH_*(X).$$

3.4. Example: matrix factorizations. Let us consider the case when $\mathcal{C} = \text{MF}(\mathbf{w})$ is the category of matrix factorizations of an isolated singularity $\mathbf{w} \in \mathfrak{m} \subset R = k[[x_1, \dots, x_n]]$, where k is a field of characteristic zero (\mathfrak{m} is the maximal ideal in R). Recall that a matrix factorization $\bar{E} = (E, \delta_E)$ of \mathbf{w} is a $\mathbb{Z}/2$ -graded finitely generated free R -module E equipped with an odd endomorphism δ such that $\delta^2 = \mathbf{w} \cdot \text{id}$. The complexes of morphisms $\mathcal{H}om(\bar{E}_1, \bar{E}_2)$ are defined by taking the spaces of $\mathbb{Z}/2$ -homogeneous R -morphisms $\text{Hom}_R^*(E_1, E_2)$ with the natural differential induced by δ_{E_1} and δ_{E_2} . Note that $\text{MF}(\mathbf{w})$ is a $\mathbb{Z}/2$ -dg-category, however, the formalism developed above works in a $\mathbb{Z}/2$ -graded case as well (cf. [10, Sec. 5.1]). We will freely use the notation and the conventions of [22, Sec. 2]. Recall that the Hochschild homology of \mathbf{w} is naturally isomorphic to the $\mathbb{Z}/2$ -graded space

$$HH_*(\text{MF}(\mathbf{w})) \simeq H(\mathbf{w}) := \Omega_{R/k}^n / d\mathbf{w} \wedge \Omega_{R/k}^{n-1}[n] \simeq (R/(\partial_1 \mathbf{w}, \dots, \partial_n \mathbf{w})) \otimes dx_1 \wedge \dots \wedge dx_n[n], \quad (3.8)$$

where we set $\partial_i = \partial/\partial x_i$ (see [10, Thm. 6.6], [22, (2.28)]).

Let

$$t : (x_1, \dots, x_n) \mapsto (t_1 x_1, \dots, t_n x_n)$$

be a diagonal symmetry of \mathbf{w} , so that $\mathbf{w}(t_1 x_1, \dots, t_n x_n) = \mathbf{w}(x_1, \dots, x_n)$. We associate with t an autoequivalence t^* of $\text{MF}(\mathbf{w})$ induced by the similar functor t^* on free $k[[x_1, \dots, x_n]]$ -modules (which acts as identity on objects and as the substitution $f \mapsto f \circ t$ on morphisms).

Recall that the kernels giving dg-functors $\mathrm{MF}(\mathbf{w}) \rightarrow \mathrm{MF}(\mathbf{w})$ can be viewed as matrix factorizations of $\tilde{\mathbf{w}} = \mathbf{w}(y_1, \dots, y_n) - \mathbf{w}(x_1, \dots, x_n)$ (see [10, Sec. 6.1], [22, Sec. 4.2]). The diagonal kernel is represented by the stabilized diagonal matrix factorization, which is the Koszul matrix factorization

$$\Delta^{st} = \Delta_{\mathbf{w}}^{st} = \{\Delta_1 \mathbf{w}, \dots, \Delta_n \mathbf{w}; y_1 - x_1, \dots, y_n - x_n\} = (R^e \otimes \bigwedge^\bullet(V), \delta^{st}) \in \mathrm{MF}(\tilde{\mathbf{w}})$$

associated with the decomposition

$$\tilde{\mathbf{w}} = \sum_{i=1}^n \Delta_i \mathbf{w}(x, y) \cdot (y_i - x_i),$$

where $R^e = R \hat{\otimes} R = k[[x_1, \dots, x_n, y_1, \dots, y_n]]$ and $V = \bigoplus_{i=1}^n k \cdot e_i$ (see [22, (2.24), (2.25)]). The endomorphism δ^{st} of the exterior algebra has two components: one is given by the exterior product with $\sum \Delta_i \mathbf{w} \cdot e_i$, and another—by the contraction with $\sum (y_i - x_i) \cdot e_i^*$.

To write explicitly the kernel corresponding to t^* we use the relation

$$F = (\mathrm{Id} \square F)(\Delta_C)$$

for $F = t^*$. The functor $(\mathrm{Id} \square t^*)$ can be identified with $(\mathrm{id} \times t)^*$, so the kernel giving t^* is the Koszul matrix factorization $(\mathrm{id} \times t)^* \Delta^{st}$. Since the functor $\mathrm{Tr} : \mathrm{MF}(\tilde{\mathbf{w}}) \rightarrow \mathrm{Per}(k)$ is given by the restriction to the diagonal $y = x$, we get an isomorphism

$$\mathrm{Tr}(t^*) \simeq ((\mathrm{id} \times t)^* \Delta^{st})|_{\Delta} \simeq \Delta^{st}|_{\Gamma_t},$$

where Γ_t is the graph of t (given by the equations $y_i = t_i x_i$). Let us renumber the coordinates in such a way that

$$t_i \neq 1 \text{ for } i \leq k \text{ and } t_{k+1} = \dots = t_n = 1.$$

Then the proof of [22, Lem. 2.5.3] gives an isomorphism

$$H^* \mathrm{Tr}(t^*) \simeq H^*(\Delta_{\Gamma_t}^{st}) \simeq H(\mathbf{w}_t), \quad (3.9)$$

where $H(\mathbf{w}_t)$ is the space (3.8) for the potential \mathbf{w}_t , the restriction of \mathbf{w} to the subspace $x_1 = \dots = x_k = 0$ of fixed points of \mathbf{w} . Note that \mathbf{w}_t still has isolated singularity (see [22, Lem. 2.5.3](i)). More explicitly, the isomorphism (3.9) is induced by the projection from the term $R^e \otimes \bigwedge^{n-k}(V)$ to $R/(x_1, \dots, x_k) \otimes e_{k+1} \wedge \dots \wedge e_n$ (which factors through the restriction to $\Delta \cap \Gamma_t$).

Now let $\bar{E} = (E, \delta_E)$ be a matrix factorization of \mathbf{w} . The computation of the generalized boundary-bulk map

$$\tau_{t^*}^{\bar{E}} : \mathrm{Hom}(\bar{E}, t^* \bar{E}) \rightarrow \mathrm{Tr}(t^*) \simeq H(\mathbf{w}_t) \quad (3.10)$$

is very similar to the computation of the equivariant Chern character in [22, Sec. 3]. By definition, we have to take the restriction to the diagonal of the morphism $c_{t^*}^{\bar{E}} = (\mathrm{id} \times t)^* c^{\bar{E}}$, where

$$c^{\bar{E}} = c_{\mathrm{Id}}^{\bar{E}} : \bar{E}^* \boxtimes \bar{E} \rightarrow \Delta^{st}$$

is the canonical morphism (2.4). The latter morphism was explicitly described in [22, Sec. 3.1, 3.2]. It is convenient to use the identification of the complex of morphisms in $\mathrm{MF}(\mathbf{w})$ from $\bar{E}^* \boxtimes \bar{E}$ to Δ^{st} with the tensor product

$$L := \Delta^{st} \otimes_{R^e} (\bar{E} \boxtimes \bar{E}^*)$$

(see [22, (3.1)]). Then the morphism $c^{\bar{E}}$ corresponds to a certain even closed element $D \in L$. Let us choose a trivialization $E \simeq U \otimes R$ of the R -module E , where U is a graded vector space. This leads to the identification

$$L \simeq \bigoplus_{j=0}^n \bigwedge^j(V) \otimes \mathrm{End}(U) \otimes R^e$$

and writing

$$D = \sum_j D_j, \quad \text{with} \quad D_j = \sum_{i_1 < \dots < i_j} e_{i_1} \wedge \dots \wedge e_{i_j} \otimes D_j(i_1, \dots, i_j)$$

with the components $D_j(i_1, \dots, i_j) \in \text{End}(U) \otimes R^e$ and $D_0 = 1 \otimes \text{id}_U$, we obtain a system of equations on these components with the unique solution such that $D_j(i_1, \dots, i_j)$ does not depend on y_r with $r < i_1$ (see [22, Lem. 3.2.1]). The morphism $(\text{id} \times t)^* c^{\bar{E}}$ corresponds to the element

$$(\text{id} \times t)^* D \in (\text{id} \times t)^* L \simeq \bigwedge^\bullet(V) \otimes \text{End}(U) \otimes R^e,$$

where the latter identification uses the action of $\text{id} \times t$ on R^e .

As the description of the isomorphism (3.9) shows, the map $\tau_{t^*}^{\bar{E}} = \text{Tr}((\text{id} \times t)^* c^{\bar{E}})$ is determined by

$$(\text{id} \times t)^* D_{n-k}(k+1, \dots, n)|_{\Delta \cap \Gamma_t} = \partial_n \delta_E(x) \circ \dots \circ \partial_{k+1} \delta_E(x)|_{x_1 = \dots = x_k = 0},$$

where we view δ_E as an element of $\text{End}(U) \otimes R$ (see the proof of [22, Thm. 3.3.3]). Thus, if use the identification $\mathcal{H}om(\bar{E}, t^* \bar{E}) \simeq (\text{End}(U) \otimes R, d)$, where d is the differential induced by δ_E , then we arrive to the formula for $\tau_{t^*}^{\bar{E}}$ similar to those of [22, Thm. 3.2.3, Thm. 3.3.3].

Proposition 3.9. *In the above situation one has*

$$\tau_{t^*}^{\bar{E}}(\alpha) = \text{str}([\partial_n \delta_E \circ \dots \circ \partial_{k+1} \delta_E \circ \alpha]|_{x_1 = \dots = x_k = 0}) \cdot dx_{k+1} \wedge \dots \wedge dx_n \mod (\partial_{k+1} \mathbf{w}_t, \dots, \partial_n \mathbf{w}_t).$$

Next, we want to compute the canonical pairing (2.6) between $\text{Tr}(F)$ and $\text{Tr}(G)$ for the autoequivalence $F = t^*$ of $\text{MF}(\mathbf{w})$. Note that in this case $G = (t^{-1})^*$, so both $\text{Tr}(F)$ and $\text{Tr}(G)$ are naturally isomorphic to the $\mathbb{Z}/2$ -graded space $H(\mathbf{w}_t) = H(\mathbf{w}_{t^{-1}})$. Again the calculation is very similar to that of the canonical pairing on the equivariant Hochschild homology in [22, Sec. 4.2]. Namely, it is enough to compute the corresponding Casimir tensor $\tau_{F,G} \in \text{Tr}(F') \otimes \text{Tr}(G)$, which is obtained by applying the generalized boundary-bulk map associated with $F' \square G$ to the canonical morphism $\Delta \rightarrow (F' \square G)(\Delta)$ (see Lemma 2.6(ii)). In our case F' is the autoequivalence $(t^{-1})^*$ of $\text{MF}(-\mathbf{w})$, so $F' \square G$ is the autoequivalence $(t^{-1} \times t^{-1})^*$ of $\text{MF}(\tilde{\mathbf{w}})$. Thus, to calculate $\tau_{F,G}$ we can apply Proposition 3.9 to the canonical isomorphism

$$\alpha : \Delta^{st} \rightarrow (t^{-1} \times t^{-1})^* \Delta^{st} \quad (3.11)$$

in the homotopy category of $\text{MF}(\tilde{\mathbf{w}})$. Note that

$$(t^{-1} \times t^{-1})^* \Delta^{st} \simeq \{(t^{-1} \times t^{-1})^* \Delta_1 \mathbf{w}, \dots, (t^{-1} \times t^{-1})^* \Delta_n \mathbf{w}; t_1^{-1}(y_1 - x_1), \dots, t_n^{-1}(y_n - x_n)\}$$

and the morphism α is characterized uniquely (up to homotopy) by the condition that the induced map on $\text{coker}(\delta_{10} : \bigwedge^1 \rightarrow \bigwedge^0)$ is identity, where δ_{10} is the component of the differential δ of the Koszul matrix factorization (see [22, Prop. 2.3.1] or [10]). We will use the following fact about Koszul matrix factorizations (that follows from the proof of [22, Lem. 2.5.5]).

Lemma 3.10. *Let $b_\bullet = (b_1, \dots, b_r)$ be a regular sequence in a local ring A , and let*

$$a_1 b_1 + \dots + a_r b_r = a'_1 b_1 + \dots + a'_r b_r = \mathbf{w} \in A$$

for some sequences a_\bullet and a'_\bullet in A . Then there exists an isomorphism of matrix factorizations

$$\{a_\bullet, b_\bullet\} \xrightarrow{\sim} \{a'_\bullet, b_\bullet\}$$

of the form $\exp(h) \wedge ?$ for some $h \in \bigwedge_A^2(A^r)$.

Let us denote by α_t the automorphism of the exterior algebra $\bigwedge^\bullet(V)$ induced by the map $t : V \rightarrow V$ sending e_i to $t_i \cdot e_i$. Then α_t gives an isomorphism of Koszul matrix factorizations

$$\alpha_t : \{t_1^{-1} f_1, \dots, t_n^{-1} f_n; (y_1 - x_1), \dots, (y_n - x_n)\} \rightarrow \{f_1, \dots, f_n; t_1^{-1}(y_1 - x_1), \dots, t_n^{-1}(y_n - x_n)\},$$

where we denoted $f_i(x, y) = (t^{-1} \times t^{-1})^* \Delta_i \mathbf{w}$. Now applying the above lemma we get an isomorphism (3.11) of the form $\alpha = \alpha_t \circ \exp(h)$. Note that α preserves the filtration

$$F_p = \bigwedge^{\geq p} \left(\bigoplus_{i=1}^r k \cdot e_i \right) \otimes \bigwedge^{\bullet} \left(\bigoplus_{j=r+1}^n k \cdot e_j \right) \otimes R^e$$

on $\bigwedge^{\bullet}(V) \otimes R^e$ (since F_p are ideals with respect to the exterior multiplication). Thus, the same argument as in [22, Sec. 4.2] shows the equality

$$\tau_{F,G} = \text{ch}(\Delta_{\mathbf{w}_t}^{st}) \cdot \det[\text{id} - t, V/V^t] = \text{ch}(\Delta_{\mathbf{w}_t}^{st}) \cdot (1 - t_1) \dots (1 - t_k),$$

where $\Delta_{\mathbf{w}_t}^{st}$ is the stabilized diagonal for the potential $\mathbf{w}_t(x_{k+1}, \dots, x_n)$. Hence, using [22, Prop. 4.1.1, 4.1.2] we arrive to the following result.

Proposition 3.11. *Under the isomorphisms $\text{Tr}(t^*) \simeq \text{Tr}((t^{-1})^*) \simeq H(\mathbf{w}_t)$ the canonical pairing $\langle \cdot, \cdot \rangle_{t^*, (t^{-1})^*}$ (see (2.6)) gets identified with*

$$\det[\text{id} - t, V/V^t]^{-1} \cdot \langle \cdot, \cdot \rangle_{\mathbf{w}_t} = (1 - t_1)^{-1} \dots (1 - t_k)^{-1} \cdot \langle \cdot, \cdot \rangle_{\mathbf{w}_t},$$

where $\langle \cdot, \cdot \rangle_{\mathbf{w}_t}$ is the canonical pairing on $H(\mathbf{w}_t) = HH_*(\text{MF}(\mathbf{w}_t))$, given by [22, (4.6)].

In the case when 0 is an isolated fixed point of t , by Propositions 3.9 and 3.11, the holomorphic Lefschetz formula (0.2) takes the form stated in Theorem 0.1.

Remark. Comparing the formula of Proposition 3.9 with that of [22, Thm. 3.3.3] we see that in the situation when G is a finite group of symmetries of \mathbf{w} , and \bar{E} is a G -equivariant matrix factorization of \mathbf{w} , the components of the boundary-bulk map $\tau^{\bar{E}}$ for the dg-category $\text{MF}_G(\mathbf{w})$ of G -equivariant matrix factorizations of \mathbf{w} are given by

$$\alpha \mapsto \tau_{g^*}^{\bar{E}}(g \circ \alpha), \quad g \in G,$$

where we use the action of $g \in G$ on E . Similarly, comparing Proposition 3.11 with the formula for the canonical pairing on the Hochschild homology of $\text{MF}_G(\mathbf{w})$, we see that the latter pairing has components

$$|G|^{-1} \langle \cdot, \cdot \rangle_{g^*, (g^{-1})^*}.$$

It seems very plausible that the same relations hold whenever a finite group G acts on a smooth and compact dg-category and we pass to the corresponding G -equivariant category.

In conclusion let us point out several simple consequences of the Lefschetz formula (0.2) in the situation considered above.

Corollary 3.12. *Let $\mathbf{w}(x_1, \dots, x_n)$ be an isolated singularity, $t \in (k^*)^n$ a diagonal symmetry of \mathbf{w} , \bar{A} and \bar{B} matrix factorizations of \mathbf{w} equipped with closed morphisms $\alpha : \bar{A} \rightarrow t^* \bar{A}$ and $\beta : t^* \bar{B} \rightarrow \bar{B}$. Assume that the fixed locus of t has odd dimension (i.e., the number of coordinates of t that are equal to 1 is odd). Then the endomorphism $(t^*, \alpha, \beta)_*$ of $\text{Hom}(\bar{A}, \bar{B})$ defined by (0.1) has zero supertrace.*

Proof. The map $\tau_{t^*}^{\bar{A}}$ takes values in the $\mathbb{Z}/2$ -graded space $\text{Tr}(t^*) \simeq H(\mathbf{w}_t)$ which lives in odd degree due to our assumption on the fixed locus of t . Hence, $\tau_{t^*}^{\bar{A}}(\alpha) = 0$ and the right-hand side of the formula (0.2) vanishes in our situation. \square

Note that Corollary 3.12 generalizes the result conjectured by Dao [8] and proved in [22] and [4] (see also [20]) that $\text{Hom}(\bar{A}, \bar{B})$ has zero Euler characteristic in the case when n is odd.

Recall that a polynomial $\mathbf{w}(x_1, \dots, x_n)$ is called *quasihomogeneous* with respect to a primitive degree vector $(d_1, \dots, d_n) \in \mathbb{Z}^n$ (where all $d_i > 0$) if

$$\mathbf{w}(\lambda^{d_1} x_1, \dots, \lambda^{d_n} x_n) = \lambda^d \mathbf{w}(x_1, \dots, x_n)$$

for some $d > 0$. In this case we can consider graded matrix factorizations \bar{A} of \mathbf{w} (\mathbb{G}_m -equivariant in terminology of [23]), i.e., $\bar{A} = (A_0, A_1, \delta)$, where A_0 and A_1 are free finite rank \mathbb{Z} -graded $k[x_1, \dots, x_n]$ -modules (where $\deg(x_i) = d_i$), the differential $\delta_0 : A_1 \rightarrow A_0$ should be homogeneous of degree 0, and the differential $\delta_1 : A_0 \rightarrow A_1$ should be of degree d .

Corollary 3.13. *Assume $\text{char}(k) = 0$.*

(i) *Assume that an isolated singularity $\mathbf{w}(x_1, \dots, x_n)$ is even, i.e., $\mathbf{w}(-x) = \mathbf{w}(x)$. Then for any matrix factorization \bar{A} of \mathbf{w} equipped with an isomorphism $(-1)^* \bar{A} \simeq \bar{A}$, the superdimension $\text{sdim}(\bar{A}|_0)^+$ of the space of $(-1)^*$ -invariants in $\bar{A}|_0$ is divisible by $2^{\lceil \frac{n}{2} \rceil - 1}$.*

(ii) *Let $\mathbf{w}(x_1, \dots, x_n)$ be a quasihomogeneous isolated singularity. For any graded matrix factorization \bar{A} of \mathbf{w} consider the Laurent polynomial*

$$h_{\bar{A}|_0}(t) = \sum_{i \in \mathbb{Z}} \text{sdim}(\bar{A}|_0)_i t^i,$$

where $(\bar{A}|_0)_i$ is the graded component of $\bar{A}|_0$ of degree i . Then $h_{\bar{A}|_0}(t)$ is divisible by $(1-t)^{\lceil \frac{n}{2} \rceil}$ in $\mathbb{Z}[t, t^{-1}]$.

Proof. (i) By Theorem 0.1, we have

$$\text{str}((-1)^*, \text{Hom}(\bar{A}, \bar{A})) = \frac{\text{str}((-1)^*, \bar{A}|_0)^2}{2^n}. \quad (3.12)$$

We have

$$\text{str}((-1)^*, \bar{A}|_0) = \text{sdim}(\bar{A}|_0)^+ - \text{sdim}(\bar{A}|_0)^-,$$

and also

$$\text{sdim}(\bar{A}|_0)^+ + \text{sdim}(\bar{A}|_0)^- = \text{sdim}(\bar{A}|_0) = 0,$$

since \mathbf{w} induces a generic isomorphism between the even and odd components of \bar{A} . Hence,

$$\text{str}((-1)^*, \bar{A}|_0) = 2 \text{sdim}(\bar{A}|_0)^+,$$

and the required divisibility follows from (3.12).

(ii) For any $t \in k^*$ let us define the endofunctor t^* of the category $\text{MF}(\mathbf{w})$ (here we consider all matrix factorizations, not necessarily graded) as the composition of the natural functor $t^* : \text{MF}(\mathbf{w}) \rightarrow \text{MF}(t^d \mathbf{w})$ with the equivalence $\text{MF}(t^d \mathbf{w}) \simeq \text{MF}(\mathbf{w})$ which sends a matrix factorization $\bar{A} = (A, \delta_0, \delta_1)$ to $(A, \delta_0, t^{-d} \delta_1)$. If \bar{A} is a graded matrix factorization then we have a natural isomorphism $\beta_t : t^* \bar{A} \rightarrow \bar{A}$ which acts by t^i on the i th graded component. Thus, $\text{str}(\beta_t, \bar{A}|_0) = h_{\bar{A}|_0}(t)$. By Theorem 0.1 (applied to $B = A$, $\alpha = \beta_t^{-1}$ and $\beta = \beta_t$), for generic $t \in k^*$ we have

$$\text{str}(t, \text{Hom}(\bar{A}, \bar{A})) = h_{\bar{A}|_0}(t^{-1}) h_{\bar{A}|_0}(t) \cdot \prod_{i=1}^n (1 - t^{d_i})^{-1},$$

where (d_1, \dots, d_n) is the degree vector making \mathbf{w} quasihomogeneous. Hence, this equality also holds for the formal variable t . Since the left-hand side is a Laurent polynomial with integer coefficients, we obtain that $h_{\bar{A}}(t^{-1}) h_{\bar{A}}(t)$ is divisible by $(1-t)^n$ in $\mathbb{Z}[t, t^{-1}]$. Since $(1-t) = -t(1-t^{-1})$, the unique factorization in $\mathbb{Z}[t, t^{-1}]$ implies that the maximal powers of $(1-t)$ dividing $h_{\bar{A}}(t)$ and $h_{\bar{A}}(t^{-1})$ are the same. The required divisibility immediately follows. \square

Remark. Assume that the quasihomogeneous polynomial \mathbf{w} can be written in the form $\mathbf{w} = a_1 b_1 + \dots + a_r b_r$, where a_i and b_i are quasihomogeneous polynomials (of positive degrees) with respect to the same degree vector (d_1, \dots, d_n) . Then one has a Koszul matrix factorization $\bar{A} = \{a_1, \dots, a_r; b_1, \dots, b_r\}$ which is a graded matrix factorization of \mathbf{w} (see [24, Def. 1.1.8]). The Laurent polynomial considered in Corollary 3.13 in this case has form

$$h_{\bar{A}|_0}(t) = (1 - t^{m_1}) \dots (1 - t^{m_r}),$$

where $m_i = \deg(b_i)$. Note that the condition that \mathbf{w} has an isolated singularity easily implies that $r \geq n/2$. Thus, the divisibility of Corollary 3.13(ii) is sharp if we can find a presentation of \mathbf{w} in the above form with $r = \lceil \frac{n}{2} \rceil$. For example, this is always the case for so called *invertible* quasihomogeneous polynomials (see [16, Sec. 1.3]).

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